

Possible Generalizations within Braneworld Scenarios: Torsion fields

J. M. Hoff da Silva*

*UNESP - Campus de Guaratinguetá - DFQ
Av. Dr. Ariberto Pereira da Cunha, 333
CEP 12516-410, Guaratinguetá - SP, Brazil*

Roldão da Rocha†

*Centro de Matemática, Computação e Cognição,
Universidade Federal do ABC, 09210-170, Santo André, SP, Brazil*

In this Chapter we introduce the aspects in which torsion can influence the formalism of braneworld scenarios in general, and also how it is possible to measure such kind of effects, namely, for instance, the blackstring transverse area corrections and variation of quasar luminosity due to those corrections. We analyze the projected effective Einstein equation in a 4-dimensional arbitrary manifold embedded in a 5-dimensional Riemann-Cartan manifold. The Israel-Darmois matching conditions are investigated, in the context where the torsion discontinuity is orthogonal to the brane. Unexpectedly, the presence of torsion terms in the connection does not modify such conditions whatsoever, despite of the modification in the extrinsic curvature and in the connection. Then, by imposing the \mathbb{Z}_2 -symmetry, the Einstein equation obtained via Gauss-Codazzi formalism is extended, in order to now encompass the torsion terms. We also show that the factors involving contorsion change drastically the effective Einstein equation on the brane, as well as the effective cosmological constant. Also, we present gravitational aspects of braneworld models endowed with torsion terms both in the bulk and on the brane. In order to investigate a conceivable and measurable gravitational effect, arising genuinely from bulk torsion terms, we analyze the variation in the black hole area by the presence of torsion. Furthermore, we extend the well known results about consistency conditions in a framework that incorporates brane torsion terms. It is shown, in a rough estimate, that the resulting effects are generally suppressed by the internal space volume. This formalism provides manageable models and their possible ramifications into some aspects of gravity in this context, and cognizable corrections and physical effects as well. The torsion influences the braneworld scenario and we can check it by developing the bulk metric Taylor expansion around the brane, which brings corrections in the blackstring transverse area. This generalization is presented in order to better probe braneworld properties in a Riemann-Cartan framework, and it is also shown how the factors involving contorsion change the effective Einstein equation on the brane, the effective cosmological constant, and their consequence in a Taylor expansion of the bulk metric around the brane.

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I. INTRODUCTION

In the last years there has been an increasing interest in large extra dimensional models [1], mainly due the developments in string theory [2] but also to the possibility of the hierarchy problem explanation, presented for instance in Randall-Sundrum braneworld scenarios [3]. In particular, the Randall-Sundrum braneworld model [3] is effectively implemented in a 5-dimensional manifold (where there is one warped extra dimension) and it is based on a 5-dimensional reduction of Hořava-Witten theory. In Randall-Sundrum models, our universe is described by an infinitely thin membrane — the brane. One attempt of explaining why gravity is so weak is by trapping the braneworld in some higher-dimensional spacetime — the bulk — wherein the brane is viewed as a submanifold. For instance, the observable universe proposed by Randall and Sundrum, in one of their two models, can be described as being a brane embedded in an AdS_5 bulk. There are several analogous models, which consider our universe as a D -dimensional braneworld embedded in a bulk of codimension one. In some models, there are some changes in the scenario that allows the presence of a compact dimension on the brane [30]. This gives rise to the so called hybrid compactification.

*Electronic address: hoff@feg.unesp.br

†Electronic address: roldao.rocha@ufabc.edu.br

As a crucial formal pre-requisite to try to describe gravity in a braneworld context, the bulk is imposed to present codimension one — in relation to the brane. There is a great amount of results applying the Gauss-Codazzi (GC) formalism [5] in order to derive the properties of such braneworld (see [6] and references therein). In the case where the bulk has two dimensions more than the brane, the GC formalism is no longer useful, since the concept of a thin membrane is meaningless, in the sense that it is not possible to define junction conditions in codimension greater than one. In such case the addition of a Gauss-Bonnet term seems to break the braneworld apparent sterility [7]. For higher codimensions, the situation is even worse.

Going back to the case of one non-compact extra dimension, after expressing the Einstein tensor in terms of the stress tensor of the bulk and extrinsic curvature corrections, it is necessary to develop some mechanism to explore some physical quantities on the brane. In order that the GC formalism be useful, we must be able to express the quantities in the limit of the extra dimension going to zero — at the point where the brane is located. Using this procedure, two junction requirements [8], which are the well known Israel-Darmois matching conditions, emerge. Moreover, torsion also emerges in the interface between GR and gravity obtained via string theory at low energy. In this vein, it seems quite natural to explore some aspects of braneworld models in the presence of torsion. This is one of the main purposes of this Chapter, where the matching conditions are analyzed and investigated in the context of a braneworld of codimension one, described by a Riemann-Cartan manifold, encoding torsion terms.

This Chapter is organized as follows: after presenting some geometric preliminaries involving Riemann-Cartan spacetimes in the Section II, in Section III the concept of torsion is introduced in the context of general relativity and the Israel-Darmois matching conditions are investigated in the presence of torsion, in a similar approach that can be found in reference [9]. In addition, junction conditions are investigated in the context where the torsion discontinuity is orthogonal to the brane. In Section IV the Gauss-Codazzi formalism is used in order to establish the role and implications of torsion terms in the braneworld framework scenario. All the quantities, like the Riemann and Ricci tensors, and the scalar curvature, and also the Einstein tensor taking into account torsion terms are written in terms of their respective partners defined in terms of the Levi-Civita connection.

In order to find some typical gravitational signatures of braneworld scenarios with torsion we obtain all the formulæ for a Taylor expansion outside a black hole in Section V, extending some results of Ref. [10] in order to encompass accrued torsion corrections. It is shown how the contortion and its derivatives determine the variation in the area of the black hole horizon along the extra dimension, inducing observable physical effects. The Taylor expansion outside a black hole metric gives information about the bulk torsion terms, where the corrections in the area of the $5D$ black string horizon are evinced.

In Section VI we apply the braneworld consistency conditions in the case when torsion is present in the brane manifold. We are particularly concerned with the viability of such an extension, analyzing the torsion effects in the brane scalar curvature. We show that, for factorizable metrics, the torsion contribution to the brane curvature is damped by the distance between the branes. In warped braneworld models, however, this damping is — at least partially — compensated by terms of the warp factor. It is also shown that if the brane manifold is endowed with a connection presenting torsion then a Randall-Sundrum like scenario with equal sign brane tension becomes possible, in acute contrast to the standard Randall-Sundrum model. Roughly speaking, this last possibility comes from the following reasoning. The presence of the torsion terms generally relax the consistence conditions, specially in what concerns the sum over the brane tensions. Therefore, the brane tensions are not restricted to the same sign, although constrained by a specific contraction of contortion terms. By studying the general consistency conditions applied to this case, we arrive at some rough estimates concerning brane torsion effects. Although the presence of torsion is not prohibited at all, its effects are generally suppressed.

II. PRELIMINARIES

A. Classification of Metric Compatible Structures (M, g, D)

Let M denote a n -dimensional manifold¹. We denote as usual by $T_x M$ and $T_x^* M$ respectively the tangent and the cotangent spaces at $x \in M$. By $TM = \bigcup_{x \in M} T_x M$ and $T^*M = \bigcup_{x \in M} T_x^* M$ respectively the tangent and cotangent bundles. The spaces $T_s^r M$ we denote the bundle of r -contravariant and s -covariant tensors and by $\mathcal{T}M = \bigoplus_{r,s=0}^{\infty} T_s^r M$ the tensor bundle. By $\bigwedge_r TM$ and $\bigwedge_r T^*M$ denote respectively the bundles of r -multivector fields and of r -form fields.

¹ We left the topology of M unspecified for while.

We call $\bigwedge TM = \bigoplus_{r=0}^{r=n} \bigwedge^r TM$ the bundle of (non homogeneous) multivector fields and call $\bigwedge T^*M = \bigoplus_{r=0}^{r=n} \bigwedge^r T^*M$ the exterior algebra (Cartan) bundle. Of course, it is the bundle of (non homogeneous) form fields. Recall that the real vector spaces are such that $\dim \bigwedge^r T_x M = \dim \bigwedge^r T_x^* M = \binom{n}{r}$ and $\dim \bigwedge T^*M = 2^n$. Some *additional* structures will be introduced or mentioned below when needed. Let² $\mathbf{g} \in \sec T_2^0 M$ a metric of signature (p, q) and D an arbitrary metric compatible connection on M , i.e., $D\mathbf{g} = 0$. We denote by \mathbf{R} and \mathbf{T} respectively the (Riemann) curvature and torsion tensors³ of the connection D , and recall that in general a given manifold given some additional conditions may admit many different metrics and many different connections.

Given a triple (M, \mathbf{g}, D) ,

- (a) it is called a Riemann-Cartan space if and only if $D\mathbf{g} = 0$, and $\mathbf{T} \neq 0$,
- (b) it is called Weyl space if and only if $D\mathbf{g} \neq 0$ and $\mathbf{T} = 0$,
- (c) it is called a Riemann space if and only if $D\mathbf{g} = 0$ and $\mathbf{T} = 0$, and in that case the pair (D, \mathbf{g}) is called *Riemannian structure*.
- (d) it is called a Riemann-Cartan-Weyl space if and only if $D\mathbf{g} \neq 0$ and $\mathbf{T} \neq 0$,
- (e) it is called Riemann flat if and only if $D\mathbf{g} = 0$ and $\mathbf{R} = 0$,
- (f) it is called teleparallel if and only if $D\mathbf{g} = 0$, $\mathbf{T} \neq 0$ and $\mathbf{R} = 0$.

B. Levi-Civita and Riemann-Cartan Connections

For each metric tensor defined on the manifold M there exists one and only one connection in the conditions of the item c) above. It is called the *Levi-Civita connection* of the metric considered, and is denoted in what follows by \hat{D} . A connection satisfying the properties in (a) above is called a Riemann-Cartan connection. In general both connections may be defined in a given manifold and they are related by well established formulas recalled below. A connection defines a rule for the parallel transport of vectors (more generally tensor fields) in a manifold, something which is conventional [11], and so the question concerning which one is more important is according to our view meaningless.

C. Spacetime Structures

When $\dim M = 4$ and the metric \mathbf{g} has signature $(1, 3)$ we sometimes substitute Riemann by Lorentz in the previous definitions (c), (e) and (f). In order to represent a spacetime structure a Lorentzian or a Riemann-Cartan structure (M, \mathbf{g}, D) need be such that M is connected and paracompact and equipped with an orientation defined by the volume element $\tau_{\mathbf{g}} \in \sec \bigwedge^4 T^*M$ and a time orientation denoted by \uparrow . We omit here the details and ask to the interested reader to consult, e.g., [12]. A general spacetime will be represented by a pentuple $(M, \mathbf{g}, D, \tau_{\mathbf{g}}, \uparrow)$.

We call in what follows Hodge bundle the quadruple $(\bigwedge T^*M, \wedge, \cdot, \tau_{\mathbf{g}})$ and now recall the meaning of the above symbols.

We suppose in what follows that any reader of this paper knows the meaning of the exterior product of form fields and its main properties⁴. We simply recall here that if $\mathcal{A}_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_s \in \sec \bigwedge^s T^*M$ then

$$\mathcal{A}_r \wedge \mathcal{B}_s = (-1)^{rs} \mathcal{B}_s \wedge \mathcal{A}_r. \quad (1)$$

Let be $\mathcal{A}_r = a_1 \wedge \dots \wedge a_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_r = b_1 \wedge \dots \wedge b_r \in \sec \bigwedge^r T^*M$ where $a_i, b_j \in \sec \bigwedge^1 T^*M$ ($i, j = 1, 2, \dots, r$).

² We denote by $\sec(X(M))$ the space of the sections of a bundle $X(M)$. Note that all functions and differential forms are supposed smooth, unless we explicitly say the contrary.

³ The precise definitions of those objects will be recalled below.

⁴ We use the conventions of [12].

(i) The scalar product $\mathcal{A}_r \cdot \mathcal{B}_r$ is defined by

$$\begin{aligned} \mathcal{A}_r \cdot \mathcal{B}_r &= (a_1 \wedge \dots \wedge a_r) \cdot (b_1 \wedge \dots \wedge b_r) \\ &= \begin{vmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_r \\ \dots & \dots & \dots \\ a_r \cdot b_1 & \dots & a_r \cdot b_r \end{vmatrix}. \end{aligned} \quad (2)$$

where $a_i \cdot b_j := \mathbf{g}(a_i, b_j)$.

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, then $\mathcal{A}_r \cdot \mathcal{B}_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\bigwedge T^*M$.

For $r \leq s$, $\mathcal{A}_r = a_1 \wedge \dots \wedge a_r$, $\mathcal{B}_s = b_1 \wedge \dots \wedge b_s$ we define the *left contraction* by

$$\lrcorner : (\mathcal{A}_r, \mathcal{B}_s) \mapsto \mathcal{A}_r \lrcorner \mathcal{B}_s = \sum_{i_1 < \dots < i_r} \epsilon^{i_1 \dots i_r} (a_1 \wedge \dots \wedge a_r) \cdot (b_{i_1} \wedge \dots \wedge b_{i_r}) \sim b_{i_r+1} \wedge \dots \wedge b_{i_s} \quad (3)$$

where \sim is the reverse mapping (*reversion*) defined by

$$\sim : \sec \bigwedge^p T^*M \ni a_1 \wedge \dots \wedge a_p \mapsto a_p \wedge \dots \wedge a_1 \quad (4)$$

and extended by linearity to all sections of $\bigwedge T^*M$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M$, $\mathcal{A}_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_s \in \sec \bigwedge^s T^*M$ then $(\alpha \mathcal{A}_r) \lrcorner \mathcal{B}_s = \mathcal{A}_r \lrcorner (\alpha \mathcal{B}_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\bigwedge T^*M$, i.e., for $\mathcal{A}, \mathcal{B} \in \sec \bigwedge T^*M$

$$\mathcal{A} \lrcorner \mathcal{B} = \sum_{r,s} \langle \mathcal{A} \rangle_r \lrcorner \langle \mathcal{B} \rangle_s, \quad r \leq s, \quad (5)$$

where $\langle \mathcal{A} \rangle_r$ means the projection of \mathcal{A} in $\bigwedge^r T^*M$.

It is also necessary to introduce the operator of *right contraction* denoted by \lrcorner . The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $\mathcal{A}_r \in \sec \bigwedge^r T^*M$, $\mathcal{B}_s \in \sec \bigwedge^s T^*M$ then $\mathcal{B}_s \lrcorner \mathcal{A}_r = (-1)^{s(r-s)} \mathcal{A}_r \lrcorner \mathcal{B}_s$.

D. Exterior derivative d and Hodge coderivative δ

The *exterior derivative* is a mapping

$$d : \sec \bigwedge T^*M \rightarrow \sec \bigwedge T^*M,$$

satisfying:

$$\begin{aligned} \text{(i)} \quad & d(A + B) = dA + dB; \\ \text{(ii)} \quad & d(A \wedge B) = dA \wedge B + \bar{A} \wedge dB; \\ \text{(iii)} \quad & df(v) = v(f); \\ \text{(iv)} \quad & d^2 = 0, \end{aligned} \quad (6)$$

for every $A, B \in \sec \bigwedge T^*M$, $f \in \sec \bigwedge^0 T^*M$ and $v \in \sec TM$.

The *Hodge codifferential* operator in the Hodge bundle is the mapping $\delta : \sec \bigwedge^r T^*M \rightarrow \sec \bigwedge^{r-1} T^*M$, given for homogeneous multiforms, by:

$$\delta = (-1)^r \star^{-1} d \star, \quad (7)$$

where \star is the Hodge star operator. The operator δ extends by linearity to all $\bigwedge T^*M$.

E. Clifford Bundles

Let (M, \mathbf{g}, ∇) be a Riemannian, Lorentzian or Riemann-Cartan structure⁵. As before let $\mathbf{g} \in \sec T_0^2 M$ be the metric on the cotangent bundle associated with $\mathbf{g} \in \sec T_2^0 M$. Then $T_x^* M \simeq \mathbb{R}^{p,q}$, where $\mathbb{R}^{p,q}$ is a vector space equipped with a scalar product $\bullet \equiv \mathbf{g}|_x$ of signature (p, q) . The Clifford bundle of differential forms $\mathcal{C}\ell(M, \mathbf{g})$ is the bundle of algebras, i.e., $\mathcal{C}\ell(M, \mathbf{g}) = \cup_{x \in M} \mathcal{C}\ell(T_x^* M, \bullet)$, where $\forall x \in M$, $\mathcal{C}\ell(T_x^* M, \bullet) = \mathbb{R}_{p,q}$, a real Clifford algebra. When the structure (M, \mathbf{g}, ∇) is part of a Lorentzian or Riemann-Cartan spacetime $\mathcal{C}\ell(T_x^* M, \bullet) = \mathbb{R}_{1,3}$ the so called *spacetime algebra*. Recall also that $\mathcal{C}\ell(M, \mathbf{g})$ is a vector bundle associated with the *g-orthonormal coframe bundle* $\mathbf{P}_{\text{SO}_{(p,q)}^\varepsilon}(M, \mathbf{g})$, i.e., $\mathcal{C}\ell(M, \mathbf{g}) = P_{\text{SO}_{(p,q)}^\varepsilon}(M, \mathbf{g}) \times_{ad} \mathbb{R}_{1,3}$ (see more details in, e.g., [12, 13]). For any $x \in M$, $\mathcal{C}\ell(T_x^* M, \bullet)$ is a linear space over the real field \mathbb{R} . Moreover, $\mathcal{C}\ell(T_x^* M)$ is isomorphic as a real vector space to the Cartan algebra $\bigwedge T_x^* M$ of the cotangent space. Then, sections of $\mathcal{C}\ell(M, \mathbf{g})$ can be represented as a *sum* of non homogeneous differential forms. Let now $\{\mathbf{e}_a\}$ be an orthonormal basis for TU and $\{\theta^a\}$ its dual basis. Then, $\mathbf{g}(\theta^a, \theta^b) = \eta^{ab}$.

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by

$$\theta^a \theta^b + \theta^b \theta^a = 2\eta^{ab} \quad (8)$$

and if $\mathcal{C} \in \mathcal{C}\ell(M, \mathbf{g})$ we have

$$\mathcal{C} = s + v_a \theta^a + \frac{1}{2!} b_{ab} \theta^a \theta^b + \frac{1}{3!} a_{abc} \theta^a \theta^b \theta^c + p \theta^{n+1}, \quad (9)$$

where $\tau_{\mathbf{g}} := \theta^{n+1} = \theta^0 \theta^1 \theta^2 \theta^3 \dots \theta^n$ is the volume element and $s, v_a, b_{ab}, a_{abc}, p \in \sec \bigwedge^0 T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$.

Let $\mathcal{A}_r, \in \sec \bigwedge^r T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g}), \mathcal{B}_s \in \sec \bigwedge^s T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec \bigwedge^1 T^* M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$,

$$a \cdot b = \frac{1}{2}(ab + ba) = \mathbf{g}(a, b). \quad (10)$$

We identify the *exterior product* ($\forall r, s = 0, 1, 2, 3$) of homogeneous forms (already introduced above) by

$$\mathcal{A}_r \wedge \mathcal{B}_s = \langle \mathcal{A}_r \mathcal{B}_s \rangle_{r+s}, \quad (11)$$

where $\langle \rangle_k$ is the *component* in $\bigwedge^k T^* M$ (projection) of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{C}\ell(M, \mathbf{g})$.

The scalar product, the left and the right are defined for homogeneous form fields that are sections of the Clifford bundle in exactly the same way as in the Hodge bundle and they are extended by linearity for all sections of $\mathcal{C}\ell(M, \mathbf{g})$.

In particular, for $\mathcal{A}, \mathcal{B} \in \sec \mathcal{C}\ell(M, \mathbf{g})$ we have

$$\mathcal{A} \lrcorner \mathcal{B} = \sum_{r,s} \langle \mathcal{A} \rangle_r \lrcorner \langle \mathcal{B} \rangle_s, \quad r \leq s. \quad (12)$$

The main formulas used in the present paper can be obtained (details may be found in [12]) from the following ones

⁵ ∇ may be the Levi-Civita connection $\overset{\circ}{D}$ of \mathbf{g} or an arbitrary Riemann-Cartan connection D .

(where $a \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(M, \mathbf{g})$):

$$\begin{aligned}
a\mathcal{B}_s &= a \lrcorner \mathcal{B}_s + a \wedge \mathcal{B}_s, \quad \mathcal{B}_s a = \mathcal{B}_s \lrcorner a + \mathcal{B}_s \wedge a, \\
a \lrcorner \mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s - (-1)^s \mathcal{B}_s a), \\
\mathcal{A}_r \lrcorner \mathcal{B}_s &= (-1)^{r(s-r)} \mathcal{B}_s \lrcorner \mathcal{A}_r, \\
a \wedge \mathcal{B}_s &= \frac{1}{2}(a\mathcal{B}_s + (-1)^s \mathcal{B}_s a), \\
\mathcal{A}_r \mathcal{B}_s &= \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|} + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|+2} + \dots + \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r+s|} \\
&= \sum_{k=0}^m \langle \mathcal{A}_r \mathcal{B}_s \rangle_{|r-s|+2k} \\
\mathcal{A}_r \cdot \mathcal{B}_r &= \mathcal{B}_r \cdot \mathcal{A}_r = \tilde{\mathcal{A}}_r \lrcorner \mathcal{B}_r = \mathcal{A}_r \lrcorner \tilde{\mathcal{B}}_r = \langle \tilde{\mathcal{A}}_r \mathcal{B}_r \rangle_0 = \langle \mathcal{A}_r \tilde{\mathcal{B}}_r \rangle_0, \\
\star \mathcal{A}_k &= \tilde{\mathcal{A}}_k \lrcorner \tau \mathbf{g} = \tilde{\mathcal{A}}_k \tau \mathbf{g}.
\end{aligned} \tag{13}$$

Two other important identities to be used below are:

$$a \lrcorner (\mathcal{X} \wedge \mathcal{Y}) = (a \lrcorner \mathcal{X}) \wedge \mathcal{Y} + \hat{\mathcal{X}} \wedge (a \lrcorner \mathcal{Y}), \tag{14}$$

for any $a \in \sec \bigwedge^1 T^*M$ and $\mathcal{X}, \mathcal{Y} \in \sec \bigwedge T^*M$, and

$$A \lrcorner (B \lrcorner C) = (A \wedge B) \lrcorner C, \tag{15}$$

for any $A, B, C \in \sec \bigwedge T^*M \hookrightarrow \mathcal{C}\ell(M, \mathbf{g})$.

F. Torsion, Curvature and Cartan Structure Equations

As we said in the beginning of Section 1 a given structure (M, \mathbf{g}) may admit many different metric compatible connections. Let then \tilde{D} be the Levi-Civita connection of \mathbf{g} and D a Riemann-Cartan connection acting on the tensor fields defined on M .

Let $U \subset M$ and consider a chart of the maximal atlas of M covering U with arbitrary coordinates $\{x^\mu\}$. Let $\{\partial_\mu\}$ be a basis for TU , $U \subset M$ and let $\{\theta^\mu = dx^\mu\}$ be the dual basis of $\{\partial_\mu\}$. The reciprocal basis of $\{\theta^\mu\}$ is denoted $\{\theta^\mu\}$, and $\mathbf{g}(\theta^\mu, \theta_\nu) := \theta^\mu \cdot \theta_\nu = \delta_\nu^\mu$.

Let also $\{\mathbf{e}_\mathbf{a}\}$ be an orthonormal basis for $TU \subset TM$ with $\mathbf{e}_\mathbf{b} = q_\mathbf{b}^\nu \partial_\nu$. The dual basis of TU is $\{\theta^\mathbf{a}\}$, with $\theta^\mathbf{a} = q_\mu^\mathbf{a} dx^\mu$. Also, $\{\theta_\mathbf{b}\}$ is the reciprocal basis of $\{\theta^\mathbf{a}\}$, i.e. $\theta^\mathbf{a} \cdot \theta_\mathbf{b} = \delta_\mathbf{b}^\mathbf{a}$. An arbitrary frame on $TU \subset TM$, coordinate or orthonormal will be denote by $\{e_\alpha\}$. Its dual frame will be denoted by $\{\vartheta^\rho\}$ (i.e., $\vartheta^\rho(e_\alpha) = \delta_\alpha^\rho$).

G. Torsion and Curvature Operators

The *torsion and curvature operators* τ and ρ of a connection D , are respectively the mappings:

$$\tau(\mathbf{u}, \mathbf{v}) = D_\mathbf{u} \mathbf{v} - D_\mathbf{v} \mathbf{u} - [\mathbf{u}, \mathbf{v}], \tag{16}$$

$$\rho(\mathbf{u}, \mathbf{v}) = D_\mathbf{u} D_\mathbf{v} - D_\mathbf{v} D_\mathbf{u} - D_{[\mathbf{u}, \mathbf{v}]}, \tag{17}$$

for every $\mathbf{u}, \mathbf{v} \in \sec TM$.

H. Torsion and Curvature Tensors

The *torsion and curvature* tensors of a connection D , are respectively the mappings:

$$\mathbf{T}(\alpha, \mathbf{u}, \mathbf{v}) = \alpha(\tau(\mathbf{u}, \mathbf{v})), \tag{18}$$

$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \alpha(\rho(\mathbf{u}, \mathbf{v})\mathbf{w}), \tag{19}$$

for every $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$.

We recall that for any differentiable functions f, g and h we have

$$\begin{aligned}\tau(g\mathbf{u}, h\mathbf{v}) &= gh\tau(\mathbf{u}, \mathbf{v}), \\ \rho(g\mathbf{u}, h\mathbf{v})f\mathbf{w} &= ghf\rho(\mathbf{u}, \mathbf{v})\mathbf{w}\end{aligned}\tag{20}$$

1. Properties of the Riemann Tensor for a Metric Compatible Connection

Note that it is quite obvious that

$$\mathbf{R}(\mathbf{w}, \alpha, \mathbf{u}, \mathbf{v}) = \mathbf{R}(\mathbf{w}, \alpha, \mathbf{v}, \mathbf{u}).\tag{21}$$

Define the tensor field \mathbf{R}' as the mapping such that for every $\mathbf{a}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \sec TM$ and $\alpha \in \sec \bigwedge^1 T^*M$.

$$\mathbf{R}'(\mathbf{w}, \mathbf{a}, \mathbf{u}, \mathbf{v}) = \mathbf{R}(\mathbf{w}, \alpha, \mathbf{v}, \mathbf{u}).\tag{22}$$

It is quite obvious that

$$\mathbf{R}'(\mathbf{w}, \mathbf{a}, \mathbf{u}, \mathbf{v}) = \mathbf{a} \cdot (\rho(\mathbf{u}, \mathbf{v})\mathbf{w}),\tag{23}$$

where

$$\alpha = \mathbf{g}(\mathbf{a}, \cdot), \quad \mathbf{a} = \mathbf{g}(\alpha, \cdot)\tag{24}$$

We now show that for any structure (M, \mathbf{g}, D) such that $D\mathbf{g} = 0$ we have for $\mathbf{c}, \mathbf{u}, \mathbf{v} \in \sec TM$,

$$\mathbf{R}'(\mathbf{c}, \mathbf{c}, \mathbf{u}, \mathbf{v}) = \mathbf{c} \cdot (\rho(\mathbf{u}, \mathbf{v})\mathbf{c}) = 0.\tag{25}$$

We start recalling that for every metric compatible connection it holds:

$$\begin{aligned}\mathbf{u}(\mathbf{v}(\mathbf{c} \cdot \mathbf{c})) &= \mathbf{u}(D_{\mathbf{v}}\mathbf{c} \cdot \mathbf{c} + \mathbf{c} \cdot D_{\mathbf{v}}\mathbf{c}) = 2\mathbf{u}(D_{\mathbf{v}}\mathbf{c} \cdot \mathbf{c}) \\ &= 2(D_{\mathbf{u}}D_{\mathbf{v}}\mathbf{c}) \cdot \mathbf{c} + 2(D_{\mathbf{u}}\mathbf{c}) \cdot D_{\mathbf{v}}\mathbf{c},\end{aligned}\tag{26}$$

Exchanging $\mathbf{u} \leftrightarrow \mathbf{v}$ in the last equation we get

$$\mathbf{v}(\mathbf{u}(\mathbf{c} \cdot \mathbf{c})) = 2(D_{\mathbf{v}}D_{\mathbf{u}}\mathbf{c}) \cdot \mathbf{c} + 2(D_{\mathbf{v}}\mathbf{c}) \cdot D_{\mathbf{u}}\mathbf{c}.\tag{27}$$

Subtracting Eq.(26) from Eq.(27) we have

$$[\mathbf{u}, \mathbf{v}](\mathbf{c} \cdot \mathbf{c}) = 2([D_{\mathbf{u}}, D_{\mathbf{v}}]\mathbf{c}) \cdot \mathbf{c}\tag{28}$$

But since

$$[\mathbf{u}, \mathbf{v}](\mathbf{c} \cdot \mathbf{c}) = D_{[\mathbf{u}, \mathbf{v}]}(\mathbf{c} \cdot \mathbf{c}) = 2(D_{[\mathbf{u}, \mathbf{v}]} \mathbf{c}) \cdot \mathbf{c},\tag{29}$$

we have from Eq.(28) that

$$([D_{\mathbf{u}}, D_{\mathbf{v}}]\mathbf{c} - D_{[\mathbf{u}, \mathbf{v}]} \mathbf{c}) \cdot \mathbf{c} = 0,\tag{30}$$

and it follows that $\mathbf{R}'(\mathbf{c}, \mathbf{c}, \mathbf{u}, \mathbf{v}) = 0$ as we wanted to show.

Prove that for any metric compatible connection,

$$\mathbf{R}'(\mathbf{c}, \mathbf{d}, \mathbf{u}, \mathbf{v}) = \mathbf{R}'(\mathbf{d}, \mathbf{c}, \mathbf{u}, \mathbf{v}).\tag{31}$$

Given an arbitrary frame $\{e_{\alpha}\}$ on $TU \subset TM$, let $\{\vartheta^{\rho}\}$ be the *dual frame*. We write:

$$\begin{aligned}[e_{\alpha}, e_{\beta}] &= c_{\alpha\beta}^{\rho} e_{\rho} \\ D_{e_{\alpha}} e_{\beta} &= \mathbf{L}_{\alpha\beta}^{\rho} e_{\rho},\end{aligned}\tag{32}$$

where $c_{\alpha\beta}^\rho$ are the *structure coefficients* of the frame $\{e_\alpha\}$ and $\mathbf{L}_{\alpha\beta}^\rho$ are the *connection coefficients* in this frame. Then, the components of the torsion and curvature tensors are given, respectively, by:

$$\begin{aligned} \mathbf{T}(\vartheta^\rho, e_\alpha, e_\beta) &= T_{\alpha\beta}^\rho = \mathbf{L}_{\alpha\beta}^\rho - \mathbf{L}_{\beta\alpha}^\rho - c_{\alpha\beta}^\rho \\ \mathbf{R}(e_\mu, \vartheta^\rho, e_\alpha, e_\beta) &= R_{\mu\alpha\beta}^\rho = e_\alpha(\mathbf{L}_{\beta\mu}^\rho) - e_\beta(\mathbf{L}_{\alpha\mu}^\rho) + \mathbf{L}_{\alpha\sigma}^\rho \mathbf{L}_{\beta\mu}^\sigma - \mathbf{L}_{\beta\sigma}^\rho \mathbf{L}_{\alpha\mu}^\sigma - c_{\alpha\beta}^\sigma \mathbf{L}_{\sigma\mu}^\rho. \end{aligned} \quad (33)$$

It is important for what follows to keep in mind the definition of the (symmetric) Ricci tensor, here denoted $\mathbf{Ric} \in \sec T_2^0 M$ and which in an arbitrary basis is written as

$$\mathbf{Ric} = R_{\mu\nu} \vartheta^\mu \otimes \vartheta^\nu := R_{\mu}{}^\rho{}_{\rho\nu} \vartheta^\mu \otimes \vartheta^\nu. \quad (34)$$

It is crucial here to take into account the *place* where the contractions in the Riemann tensor takes place according to our conventions.

We also have:

$$\begin{aligned} d\vartheta^\rho &= -\frac{1}{2} c_{\alpha\beta}^\rho \vartheta^\alpha \wedge \vartheta^\beta, \\ D_{e_\alpha} \vartheta^\rho &= -\mathbf{L}_{\alpha\beta}^\rho \vartheta^\beta, \end{aligned} \quad (35)$$

where $\omega_\beta^\rho \in \sec \bigwedge^1 T^* M$ are the *connection 1-forms*, $\mathbf{L}_{\alpha\beta}^\rho$ are said to be the connection coefficients in the given basis, and the $\mathcal{T}^\rho \in \sec \bigwedge^2 T^* M$ are the *torsion 2-forms* and the $\mathcal{R}_\beta^\rho \in \sec \bigwedge^2 T^* M$ are the *curvature 2-forms*, given by:

$$\begin{aligned} \omega_\beta^\rho &= \mathbf{L}_{\alpha\beta}^\rho \vartheta^\alpha, \\ \mathcal{T}^\rho &= \frac{1}{2} T_{\alpha\beta}^\rho \vartheta^\alpha \wedge \vartheta^\beta \\ \mathcal{R}_\mu^\rho &= \frac{1}{2} R_{\mu\alpha\beta}^\rho \vartheta^\alpha \wedge \vartheta^\beta. \end{aligned} \quad (36)$$

Multiplying Eqs.(33) by $\frac{1}{2} \vartheta^\alpha \wedge \vartheta^\beta$ and using Eqs.(35) and (36), we get the so-called Cartan Structure Equations:

$$\begin{aligned} d\vartheta^\rho + \omega_\beta^\rho \wedge \vartheta^\beta &= \mathcal{T}^\rho, \\ d\omega_\mu^\rho + \omega_\beta^\rho \wedge \omega_\mu^\beta &= \mathcal{R}_\mu^\rho. \end{aligned} \quad (37)$$

We can show that the torsion and (Riemann) curvature tensors can be written as

$$\mathbf{T} = e_\alpha \otimes \mathcal{T}^\alpha, \quad (38)$$

$$\mathbf{R} = e_\rho \otimes e^\mu \otimes \mathcal{R}_\mu^\rho. \quad (39)$$

I. Exterior Covariant Derivative \mathbf{D}

Sometimes, Eqs.(37) are written by some authors [14] as:

$$\mathbf{D}\vartheta^\rho = \mathcal{T}^\rho, \quad (40)$$

$$“\mathbf{D}\omega_\mu^\rho = \mathcal{R}_\mu^\rho.” \quad (41)$$

and $\mathbf{D} : \sec \bigwedge T^* M \rightarrow \sec \bigwedge T^* M$ is said to be the *exterior covariant derivative* related to the connection D . Now, Eq.(41) has been printed with quotation marks due to the fact that it is an *incorrect* equation. Indeed, a *legitimate* exterior covariant derivative operator⁶ is a concept that can be defined for $(p+q)$ -indexed r -form fields⁷ as follows. Suppose that $X \in \sec T_p^{r+q} M$ and let

$$X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} \in \sec \bigwedge^r T^* M, \quad (42)$$

⁶ Sometimes also called exterior covariant differential.

⁷ Which is not the case of the connection 1-forms ω_β^α , despite the name. More precisely, the ω_β^α are not true indexed forms, i.e., there does not exist a tensor field ω such that $\omega(e_i, e_\beta, \vartheta^\alpha) = \omega_\beta^\alpha(e_i)$.

such that for $v_i \in \sec TM$, $i = 0, 1, 2, \dots, r$,

$$X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p}(v_1, \dots, v_r) = X(v_1, \dots, v_r, e_{\nu_1}, \dots, e_{\nu_q}, \vartheta^{\mu_1}, \dots, \vartheta^{\mu_p}). \quad (43)$$

The exterior covariant differential \mathbf{D} of $X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p}$ on a manifold with a general connection D is the mapping:

$$\mathbf{D} : \sec \bigwedge^r T^*M \rightarrow \sec \bigwedge^{r+1} T^*M, \quad 0 \leq r \leq 4, \quad (44)$$

such that⁸

$$\begin{aligned} & (r+1)\mathbf{D}X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p}(v_0, v_1, \dots, v_r) \\ &= \sum_{\nu=0}^r (-1)^\nu D_{\mathbf{e}_\nu} X(v_0, v_1, \dots, \check{v}_\nu, \dots, v_r, e_{\nu_1}, \dots, e_{\nu_q}, \vartheta^{\mu_1}, \dots, \vartheta^{\mu_p}) \\ &- \sum_{0 \leq \lambda, \varsigma \leq r} (-1)^{\nu+\varsigma} X(\mathbf{T}(v_\lambda, v_\varsigma), v_0, v_1, \dots, \check{v}_\lambda, \dots, \check{v}_\varsigma, \dots, v_r, e_{\nu_1}, \dots, e_{\nu_q}, \vartheta^{\mu_1}, \dots, \vartheta^{\mu_p}). \end{aligned} \quad (45)$$

Then, we may verify that

$$\begin{aligned} \mathbf{D}X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} &= dX_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} + \omega_{\mu_s}^{\mu_1} \wedge X_{\nu_1 \dots \nu_q}^{\mu_s \dots \mu_p} + \dots + \omega_{\mu_s}^{\mu_1} \wedge X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p} \\ &- \omega_{\nu_1}^{\nu_s} \wedge X_{\nu_s \dots \nu_q}^{\mu_1 \dots \mu_p} - \dots - \omega_{\nu_s}^{\mu_1} \wedge X_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_p}. \end{aligned} \quad (46)$$

Note that if Eq.(46) is applied on any one of the connection 1-forms ω_ν^μ we would get $\mathbf{D}\omega_\nu^\mu = d\omega_\nu^\mu + \omega_\alpha^\mu \wedge \omega_\nu^\alpha - \omega_\nu^\alpha \wedge \omega_\alpha^\mu$.

III. JUNCTION CONDITIONS

In this Section some mathematical preliminaries — necessary to investigate braneworld junction conditions in a D -dimensional Riemann-Cartan manifold, embedded in an arbitrary $(D+1)$ -dimensional manifold — are briefly presented and discussed. For a complete exposition concerning arbitrary manifolds and fiber bundles, see, e.g., [15–19].

Hereon Σ denotes a D -dimensional Riemann-Cartan manifold modeling a brane embedded in a bulk, denoted by M . A vector space endowed with a constant signature metric, isomorphic to \mathbb{R}^{D+1} , can be identified at a point $x \in M$ as being the space $T_x M$ tangent to M , where M is locally diffeomorphic to its own (local) foliation $\mathbb{R} \times \Sigma$. There always exists a 1-form field n , normal to Σ , which can be locally interpreted — in the case where n is timelike — as being cotangent to the worldline of observer families, i.e., the dual reference frame relative velocity associated with such observers.

Denote $\{e_a\}$ ($a = 0, 1, \dots, D$) a basis for the tangent space $T_x \Sigma$ at a point x in Σ , and naturally the cotangent space at $x \in \Sigma$ has an orthonormal basis $\{e^a\}$ such that $e^a(e_b) = \delta_b^a$. A reference frame at an arbitrary point in the bulk is denoted by $\{e_\alpha\}$ ($\alpha = 0, 1, 2, \dots, D+1$). When a local coordinate chart is chosen, it is possible to represent $e_\alpha = \partial/\partial x^\alpha \equiv \partial_\alpha$ and $e^\alpha = dx^\alpha$. The 1-form field orthogonal to the sections of $T\Sigma$ — the tangent bundle of Σ — can now be written as $n = n_\alpha e^\alpha$, and consider the Gaussian coordinate ℓ orthogonal to the section of $T\Sigma$, indicating how much an observer move out the D -dimensional brane into the $(D+1)$ -dimensional bulk. A vector field $v = v^\alpha e_\alpha$ in the bulk is split in components in the brane and orthogonal to the brane, respectively as $v = v^a e_a + \ell e_{D+1}$. Since the bulk is endowed with a non-degenerate bilinear symmetric form g that can be written in a coordinate basis as $g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$, the components of the metric in the brane and on the bulk are denoted respectively by $q_{\alpha\beta}$ and $g_{\alpha\beta}$, and related by

$$g_{\alpha\beta} = q_{\alpha\beta} + n_\alpha n_\beta. \quad (47)$$

The 1-form field n orthogonal to Σ , in the direction of increasing ℓ is given by $n = (\partial_\alpha \ell) e^\alpha$, and its covariant components are explicitly given by $n_\alpha = \partial_\alpha \ell$. Without loss of generality a timelike hypersurface Σ is taken, where a congruence of geodesics goes across it. Denoting the proper distance (or proper time) along these geodesics by ℓ , it is always possible to put $\ell = 0$ on Σ .

⁸ As usual the inverted hat over a symbol (in Eq.(45)) means that the corresponding symbol is missing in the expression.

Denoting $\{x^\alpha\}$ a chart on both sides of the brane, define another chart $\{y^a\}$ on the brane. Here the same notation used in [9] is adopted, where Latin indices is used for hypersurface coordinates and Greek indices for coordinates in the embedding spacetime. The brane can be parametrized by $x^\alpha = x^\alpha(y^a)$, and the terms $h_a^\alpha := \frac{\partial x^\alpha}{\partial y^a}$ satisfy $h_a^\alpha n_\alpha = 0$. For displacements on the brane, it follows that

$$\begin{aligned} g &= g_{\alpha\beta} dx^\alpha \otimes dx^\beta = g_{\alpha\beta} \left(\frac{\partial x^\alpha}{\partial y^a} dy^a \right) \otimes \left(\frac{\partial x^\beta}{\partial y^b} dy^b \right) \\ &= q_{ab} dy^a \otimes dy^b, \end{aligned} \quad (48)$$

and so the induced metric components q_{ab} on Σ is related to $g_{\alpha\beta}$ by $q_{ab} = g_{\alpha\beta} h_a^\alpha h_b^\beta$.

Denoting by $[A] = \lim_{\ell \rightarrow 0^+} (A) - \lim_{\ell \rightarrow 0^-} (A)$ the change in a differential form field A across the braneworld Σ (wherein $\ell = 0$), the continuity of the chart x^α and ℓ across Σ implies that n_α and h_a^α are continuous, or, equivalently, $[n_\alpha] = [h_a^\alpha] = 0$.

Now, using the Heaviside distribution $\Theta(\ell)$ properties⁹

$$\Theta^2(\ell) = \Theta(\ell), \quad \Theta(\ell)\Theta(-\ell) = 0, \quad \frac{d}{d\ell} \Theta(\ell) = \delta(\ell),$$

the metric components $g_{\alpha\beta}$ can be written as distribution-valued tensor components

$$g_{\alpha\beta} = \Theta(\ell) g_{\alpha\beta}^+ + \Theta(-\ell) g_{\alpha\beta}^-,$$

where $g_{\alpha\beta}^+$ ($g_{\alpha\beta}^-$) denotes the metric on the $\ell > 0$ ($\ell < 0$) side of Σ . Differentiating the above expression, it reads

$$\partial_\gamma g_{\alpha\beta} = \Theta(\ell) \partial_\gamma g_{\alpha\beta}^+ + \Theta(-\ell) \partial_\gamma g_{\alpha\beta}^- + \delta(\ell) [g_{\alpha\beta}] n_\gamma.$$

It can be shown that the condition $[g_{\alpha\beta}] = 0$ must be imposed for the connection to be defined as a distribution¹⁰, also implying the ‘first’ junction condition $[h_{ab}] = 0$.

Besides a curvature associated with the connection that endows the bulk, in a Riemann-Cartan manifold the torsion associated with the connection is in general non zero. Its components can be written in terms of the connection components $\Gamma^\rho_{\beta\alpha}$ as

$$T^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} - \Gamma^\rho_{\alpha\beta}. \quad (49)$$

The general connection components is related to the Levi-Civita connection components $\overset{\circ}{\Gamma}^\rho_{\alpha\beta}$ — associated with the spacetime metric $g_{\alpha\beta}$ components — through $\Gamma^\rho_{\alpha\beta} = \overset{\circ}{\Gamma}^\rho_{\alpha\beta} + K^\rho_{\alpha\beta}$, where $K^\rho_{\alpha\beta} = \frac{1}{2} (T_\alpha^\rho{}_\beta + T_\beta^\rho{}_\alpha - T^\rho_{\alpha\beta})$ denotes the contortion tensor components. It must be emphasized that curvature and torsion are properties of a connection, not of spacetime. For instance, the Christoffel and the general connections present different curvature and torsion, although they endow the very same manifold.

The easiest method of introducing torsion terms in the theory is via the addition of an antisymmetric part in the affine connection. The general connection components are related to the Levi-Civita connection components $\overset{\circ}{\Gamma}^\rho_{\alpha\beta}$ — associated with the spacetime metric $g_{\alpha\beta}$ components — through $\Gamma^\rho_{\alpha\beta} = \overset{\circ}{\Gamma}^\rho_{\alpha\beta} + K^\rho_{\alpha\beta}$, where $K^\rho_{\alpha\beta} = \frac{1}{2} (T_\alpha^\rho{}_\beta + T_\beta^\rho{}_\alpha - T^\rho_{\alpha\beta})$ denotes the contortion tensor components. Hereon the quantities denoted by $\overset{\circ}{X}$ are constructed with the usual metric compatible torsionless Levi-Civita connection components $\overset{\circ}{\Gamma}^\rho_{\alpha\beta}$. We remark that the source of contortion may be considered as the rank-2 antisymmetric potential Kalb-Ramond (KR) field $B_{\alpha\beta}$, arising as a massless mode in heterotic string theories [20, 21]. Hereon we shall consider the formal geometric contortion, although the contortion induced by the KR field can be considered in the 5-dimensional formalism when the prescription $K^\rho_{\alpha\beta} = -\frac{1}{M^{3/2}} H^\rho_{\alpha\beta}$ is taken into account, where $H_{\rho\alpha\beta} = \partial_{[\rho} B_{\alpha\beta]}$ and M denotes the 5-dimensional Planck mass. The identification between the KR field and the contortion can be always taken into account when necessary, depending on the physical aspect of the formalism that must be emphasized, although the formalism is not precisely concerned with the fount of contortion, but with its consequences.

⁹ $\delta(\ell)$ is the Dirac distribution.

¹⁰ Basically, if the condition $[g_{\alpha\beta}] = 0$ is not imposed, there appears the product $\Theta\delta$, which is not well defined in the Levi-Civita part of the connection.

Now the distribution-valued Riemann tensor is calculated, in order to find the ‘second’ junction condition — the Israel matching condition. From the Christoffel symbols, it reads $\Gamma_{\beta\gamma}^\alpha = \Theta(\ell)\Gamma_{\beta\gamma}^{+\alpha} + \Theta(-\ell)\Gamma_{\beta\gamma}^{-\alpha}$, where $\Gamma_{\beta\gamma}^{\pm\alpha}$ are the Christoffel symbols obtained from $g_{\alpha\beta}^\pm$. Thus

$$\partial_\delta \Gamma_{\beta\gamma}^\alpha = \Theta(\ell)\partial_\delta \Gamma_{\beta\gamma}^{+\alpha} + \Theta(-\ell)\partial_\delta \Gamma_{\beta\gamma}^{-\alpha} + \delta(\ell)[\Gamma_{\beta\gamma}^\alpha]n_\delta,$$

and the Riemann tensor is given by $R_{\beta\gamma\delta}^\alpha = \Theta(\ell)R_{\beta\gamma\delta}^{+\alpha} + \Theta(-\ell)R_{\beta\gamma\delta}^{-\alpha} + \delta(\ell)A_{\beta\gamma\delta}^\alpha$, where $A_{\beta\gamma\delta}^\alpha = [\Gamma_{\beta\delta}^\alpha]n_\gamma - [\Gamma_{\beta\gamma}^\alpha]n_\delta$ [9].

The next step is to find an explicit expression for the tensor $A_{\beta\gamma\delta}^\alpha$. Observe that the continuity of the metric across Σ implies that the tangential derivatives of the metric must be also continuous. If $\partial_\gamma g_{\alpha\beta} \equiv g_{\alpha\beta,\gamma}$ is indeed discontinuous, this discontinuity must be directed along the normal vector n^α . It is therefore possible to write

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta}n_\gamma,$$

for some tensor $\kappa_{\alpha\beta}$ (given explicitly by $\kappa_{\alpha\beta} = [g_{\alpha\beta,\gamma}]n^\gamma$). Then it follows that

$$[\overset{\circ}{\Gamma}_{\beta\gamma}^\alpha] = \frac{1}{2}(\kappa_\beta^\alpha n_\gamma + \kappa_\gamma^\alpha n_\beta - \kappa_{\beta\gamma}n^\alpha),$$

and supposing that the discontinuity in the torsion terms obey the same rule as the discontinuity of $[g_{\alpha\beta,\gamma}]$, i. e., that $[T_{\beta\gamma}^\alpha] = \zeta_\beta^\alpha n_\gamma$, it reads

$$[K_{\beta\gamma}^\alpha] = \frac{1}{2}(\zeta_\beta^\alpha n_\gamma + \zeta_\gamma^\alpha n_\beta - \zeta_{\beta\gamma}n^\alpha). \quad (50)$$

The components $\kappa_{\rho\sigma}$ emulate an intrinsic property of the brane itself. The torsion is continuous along the brane, and if there is some discontinuity, it is proportional to the extra dimension. Such proportionality is given, in principle, by another quantity ζ_β^α related to the brane. After these considerations, it follows that

$$[\Gamma_{\beta\gamma}^\alpha] = \frac{1}{2}((\kappa_\beta^\alpha + \zeta_\beta^\alpha - \zeta_\beta^\alpha)n_\gamma + (\kappa_\gamma^\alpha + \zeta_\gamma^\alpha)n_\beta - \kappa_{\beta\gamma}n^\alpha),$$

and hence

$$\begin{aligned} A_{\beta\gamma\delta}^\alpha &= \frac{1}{2}(\kappa_\delta^\alpha n_\beta n_\gamma - \kappa_\gamma^\alpha n_\beta n_\delta - \kappa_{\beta\delta}n^\alpha n_\gamma + \kappa_{\beta\gamma}n^\alpha n_\delta) \\ &\quad + \frac{1}{2}(\zeta_\delta^\alpha n_\beta n_\gamma - \zeta_\gamma^\alpha n_\beta n_\delta). \end{aligned} \quad (51)$$

Denoting $\kappa = \kappa_\alpha^\alpha$ and $\zeta = \zeta_\beta^\beta$, and suitably contracting two indices, it reads

$$\begin{aligned} A_{\beta\delta} &= \frac{1}{2}(\kappa_\delta^\alpha n_\beta n_\alpha - \kappa n_\beta n_\delta - \kappa_{\beta\delta} + \kappa_{\beta\alpha}n^\alpha n_\delta) \\ &\quad + \frac{1}{2}(\zeta_\delta^\alpha n_\beta n_\alpha - \zeta n_\beta n_\delta), \end{aligned} \quad (52)$$

and also

$$A = g^{\beta\delta}A_{\beta\delta} = (\kappa_\alpha n^\alpha n^\alpha - \kappa) + \frac{1}{2}(\zeta_\alpha n^\alpha n^\alpha - \zeta).$$

The δ -function part of the Einstein tensor $G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is given by

$$\begin{aligned} S_{\beta\delta} &= A_{\beta\delta} - \frac{1}{2}g_{\beta\delta}A \\ &= \frac{1}{2}(\kappa_\delta^\alpha n_\beta n_\alpha - \kappa n_\beta n_\delta - \kappa_{\beta\delta} + \kappa_{\beta\alpha}n^\alpha n_\delta \\ &\quad - g_{\beta\delta}(\kappa_{\rho\sigma}n^\rho n^\sigma - \kappa)) + \frac{1}{2}(\zeta_\delta^\alpha n_\beta n_\alpha - \zeta n_\beta n_\delta) \\ &\quad - \frac{1}{4}g_{\beta\delta}(\zeta_{\rho\sigma}n^\rho n^\sigma - \zeta). \end{aligned} \quad (53)$$

On the other hand, the total stress-energy tensor is of the form

$$\pi_{\alpha\beta}^{\text{total}} = \Theta(\ell)\pi_{\alpha\beta}^+ + \Theta(-\ell)\pi_{\alpha\beta}^- + \delta(\ell)\pi_{\alpha\beta},$$

where $\pi_{\alpha\beta}^+$ and $\pi_{\alpha\beta}^-$ represent the bulk stress-energy in the regions where $\ell > 0$ and $\ell < 0$ respectively, while $\pi_{\alpha\beta}$ denotes the stress-energy localized on the hypersurface Σ itself. From the Einstein equations, it follows that $\pi_{\alpha\beta} = (G_N)^{-1}S_{\alpha\beta}$.

Note that, since $\pi_{\alpha\beta}$ is tangent to the brane, it follows that $\pi_{\alpha\beta}n^\beta = 0$. However, from Eq.(53) the following equation

$$\begin{aligned} 4G_N\pi_{\alpha\beta}n^\beta &= \frac{1}{2}(\zeta_{\rho\sigma}n^\rho n^\sigma - \zeta)n_\alpha \\ &= -\frac{1}{2}\zeta_{\rho\sigma}q^{\rho\sigma}n_\alpha, \end{aligned} \quad (54)$$

is derived, which means that, in order to keep the consistence of the formalism, one has to impose $\zeta_{\rho\sigma}q^{\rho\sigma} = 0$, and the last term of Eq.(53) vanishes. Note that $\pi_{\alpha\beta}$ can be expressed by $\pi_{ab} = \pi_{\alpha\beta}h_a^\alpha h_b^\beta$, just using the h_a^α vierbein introduced in the previous Section. So, taking into account that $\pi_{\alpha\beta} = (G_N)^{-1}S_{\alpha\beta}$ and Eq.(53), it reads [9]

$$4G_N\pi_{ab} = -\kappa_{\alpha\beta}h_a^\alpha h_b^\beta + q^{rs}\kappa_{\mu\nu}h_r^\mu h_s^\nu q_{ab}. \quad (55)$$

Finally, relating the jump in the extrinsic curvature to $\kappa_{\rho\sigma}$, via the covariant derivative associated to $q_{\alpha\beta}$, the following expression can be obtained from Eq.(50):

$$\begin{aligned} [\nabla_\alpha n_\beta] &= \frac{1}{2}(\kappa_{\alpha\beta} - \kappa_{\gamma\alpha}n_\beta n^\gamma - \kappa_{\gamma\beta}n_\alpha n^\gamma) \\ &+ \frac{1}{2}(\zeta_\alpha^\gamma n_\beta + \zeta_\beta^\gamma n_\alpha - \zeta_\alpha^\gamma n_\beta)n_\gamma. \end{aligned} \quad (56)$$

However, it is clear that this jump of the extrinsic curvature across the brane, $[\nabla_\alpha n_\beta] \equiv [\Xi_{\alpha\beta}]$, can be also decomposed in terms of h_a^α vectors, leading to

$$[\Xi_{ab}] = \frac{1}{2}\kappa_{\alpha\beta}h_a^\alpha h_b^\beta. \quad (57)$$

Hence, after all, it follows that

$$2G_N\pi_{ab} = -[\Xi_{ab}] + [\Xi]q_{ab}. \quad (58)$$

It means that the second matching condition is absolutely the same that is valid without any torsion term.

Once investigated the matching conditions in the presence of torsion terms, and under the assumptions of discontinuity across the brane, both the junctions conditions are shown to be the same as the usual case ($\Gamma_{\alpha\beta}^\rho = \overset{\circ}{\Gamma}_{\alpha\beta}^\rho$). We remark that, since the covariant derivative changes by torsion, the extrinsic curvature is also modified, and then the conventional arguments point in the direction of some modification in the matching conditions. However, it seems that the rôle of torsion terms in the braneworld picture is restricted to the geometric part of effective Einstein equation on the brane. More explicitly, looking at the equation that relates the Einstein equation in four dimensions with bulk quantities (see, for example [6]) we have

$$\begin{aligned} {}^{(4)}G_{\rho\sigma} &= \frac{2k_5^2}{3}\left(T_{\alpha\beta}q_\rho^\alpha q_\sigma^\beta + (T_{\alpha\beta}n^\alpha n^\beta - \frac{1}{4}T)q_{\rho\sigma}\right) \\ &+ \Xi\Xi_{\rho\sigma} - \Xi_\rho^\alpha \Xi_{\alpha\sigma} - \frac{1}{2}q_{\rho\sigma}(\Xi^2 - \Xi^{\alpha\beta}\Xi_{\alpha\beta}) \\ &- {}^{(4)}C_{\beta\gamma\epsilon}^\alpha n_\alpha n^\gamma q_\rho^\beta q_\sigma^\epsilon, \end{aligned} \quad (59)$$

where $T_{\rho\sigma}$ denotes the energy-momentum tensor, $\Xi_{\rho\sigma} = q_\rho^\alpha q_\sigma^\beta \nabla_\alpha n_\beta$ is the extrinsic curvature, k_5 denotes the 5-dimensional gravitational constant, and ${}^{(5)}C_{\beta\rho\sigma}^\alpha$ denotes the Weyl tensor. By restricting to quantities evaluated on the brane, or tending to the brane, we see that the only way to get some contribution from torsion terms is via the term ${}^{(4)}G_{\rho\sigma}$, and also via the Weyl tensor. It does not intervene in the extrinsic curvature tending to the brane.

IV. TORSION INFLUENCE ON THE PROJECTED EQUATIONS ON THE BRANE

In order to explicit the influence of contorsion terms in the projected equations on the brane, we shall to complete the GC program, from five to four dimensions, to the case with torsion. Note the by imposing the \mathbb{Z}_2 -symmetry, the extrinsic curvature reads

$$\Xi_{\alpha\beta}^+ = -\Xi_{\alpha\beta}^- = -2G_N \left(\pi_{\alpha\beta} - \frac{q_{\alpha\beta}\pi_\gamma^\gamma}{4} \right), \quad (60)$$

in such way that Eq.(58) reads¹¹

$$\Xi_{\alpha\beta} = -G_N \left(\pi_{\alpha\beta} - \frac{q_{\alpha\beta}\pi_\gamma^\gamma}{4} \right). \quad (61)$$

Decomposing the stress-tensor associated with the bulk in $T_{\alpha\beta} = -\Lambda g_{\alpha\beta} + \delta S_{\alpha\beta}$ and $S_{\alpha\beta} = -\lambda q_{\alpha\beta} + \pi_{\alpha\beta}$, where Λ is the bulk cosmological constant and λ the brane tension, and substituting into Eq.(59) it follows after some algebra¹²,

$${}^{(4)}G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - E_{\mu\nu}, \quad (62)$$

where $E_{\mu\nu} = {}^{(5)}C_{\beta\gamma\sigma}^\alpha n_\alpha n^\gamma q_\mu^\beta q_\nu^\sigma$ encodes the Weyl tensor contribution, $G_N = \frac{\lambda k_5^4}{48\pi}$ is the analogous of the Newton gravitational constant, the tensor $Y_{\mu\nu}$ is quadratic in the brane stress-tensor and given by $Y_{\mu\nu} = -\frac{1}{4}\pi_{\mu\alpha}\pi_\nu^\alpha + \frac{1}{12}\pi_\gamma^\gamma\pi_{\mu\nu} + \frac{1}{8}q_{\mu\nu}\pi_{\alpha\beta}\pi^{\alpha\beta} - \frac{1}{2}q_{\mu\nu}(\pi_\gamma^\gamma)^2$ and $\Lambda_4 = \frac{k_5^2}{2} \left(\Lambda + \frac{1}{6}k_5^2\lambda^2 \right)$ is the effective brane cosmological constant.

It is well known that the Riemann and Ricci tensors, and the curvature scalar written in terms of torsion are related with their partners, constructed with the usual metric compatible Levi-Civita connection by

$$R_{\tau\alpha\beta}^\lambda = \mathring{R}_{\tau\alpha\beta}^\lambda + \nabla_\alpha K_{\tau\beta}^\lambda - \nabla_\beta K_{\tau\alpha}^\lambda + K_{\gamma\alpha}^\lambda K_{\tau\beta}^\gamma - K_{\gamma\beta}^\lambda K_{\tau\alpha}^\gamma, \quad (63)$$

$$R_{\tau\beta} = \mathring{R}_{\tau\beta} + \nabla_\lambda K_{\tau\beta}^\lambda - \nabla_\beta K_{\tau\lambda}^\lambda + K_{\gamma\lambda}^\lambda K_{\tau\beta}^\gamma - K_{\tau\gamma}^\lambda K_{\lambda\beta}^\gamma \quad (64)$$

and

$$R = \mathring{R} + 2\nabla^\lambda K_{\lambda\tau}^\tau - K_{\tau\lambda}^\lambda K^{\tau\lambda}_\lambda + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma}, \quad (65)$$

where the quantities \mathring{X} are constructed with the usual metric compatible Levi-Civita connection, and ∇ denotes the covariant derivative *without* torsion. Clearly such relations holds in any dimension. Therefore, by denoting D_μ the covariant 4-dimensional derivative acting on the brane, it is easy to see that, from Eqs.(63),(64), and (65), the Einstein tensor on the brane is given by

$$\begin{aligned} {}^{(4)}G_{\mu\nu} = & {}^{(4)}\mathring{G}_{\mu\nu} + D_\lambda {}^{(4)}K_{\mu\nu}^\lambda - D_\nu {}^{(4)}K_{\mu\lambda}^\lambda + {}^{(4)}K_{\gamma\lambda}^\lambda {}^{(4)}K_{\mu\nu}^\gamma - {}^{(4)}K_{\mu\gamma}^\lambda {}^{(4)}K_{\lambda\nu}^\gamma - q_{\mu\nu} \left(D^\lambda {}^{(4)}K_{\lambda\tau}^\tau + \frac{1}{2} {}^{(4)}K_{\tau\lambda}^\lambda {}^{(4)}K^{\tau\gamma}_\gamma \right. \\ & \left. + \frac{1}{2} {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\gamma\lambda} \right). \end{aligned} \quad (66)$$

Note the appearance of terms multiplying the brane metric. As it shall be seen, these terms compose a new effective cosmological constant.

The presence of extra dimensions seems to be an almost inescapable characteristic of high-energy physics based upon the auspices of string theory. In this context, specific string theory inspired scenarios, in which our universe is modeled by a brane — the braneworld scenario — acquired special attention [3] due to the possibility of solving the hierarchy problem. Concomitantly, the presence of torsion is also an output of string theory [20]. Indeed, when gravitation is recovered from string theory, a plenty of physical fields abound, including the torsion field. In this context, among other motivations, it seems natural to explore some properties of braneworld models in the presence of torsion.

¹¹ Hereon, we remove the + and - labels.

¹² See, please, reference [6] for all the details.

V. MEASURABLE TORSION EFFECTS

In the previous Section we proved that although the presence of torsion terms in the connection does not modify the Israel-Darmois matching conditions. The factors involving contortion alter drastically the effective Einstein equation on the brane, and also the function involving contortion terms that is analogous to the effective cosmological constant as well.

We shall use such results to extend the bulk metric Taylor expansion in terms of the brane metric, in a direction orthogonal to the brane, encompassing torsion terms. As an immediate application, the corrections in a black hole horizon area due to contortion terms are achieved.

Using the Einstein tensor on the brane encoding torsion terms, the $E_{\mu\nu}$ tensor can be expressed in terms of the bulk contortion terms by

$$E_{\kappa\delta} = \mathring{E}_{\kappa\delta} + \left(\nabla_{[\nu} K^\mu_{\alpha\beta]} + K^\mu_{\gamma[\nu} K^\gamma_{\alpha\beta]} \right) n_\mu n^\nu q_\kappa^\alpha q_\delta^\beta - \frac{2}{3} (q_\kappa^\alpha q_\delta^\beta + n^\alpha n^\beta q_{\kappa\delta}) \left(\nabla_{[\lambda} K^\lambda_{\beta\alpha]} + K^\lambda_{\gamma\lambda} K^\gamma_{\beta\alpha} - K^\sigma_{\beta\gamma} K^\gamma_{\sigma\alpha} \right) + \frac{1}{6} q_{\kappa\delta} \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right), \quad (67)$$

where ∇_μ is the bulk covariant derivative. Now, the explicit influence of the contortion terms in the Einstein brane equation can be visualized. From Eqs.(62), (67) and expressing the torsion terms of the Einstein brane tensor (see Eq. (20) of reference [22]), it follows that

$$\begin{aligned} & {}^{(4)}\hat{G}_{\mu\nu} + D_{[\lambda} {}^{(4)}K^\lambda_{\mu\nu]} + {}^{(4)}K^\delta_{\gamma\delta} {}^{(4)}K^\lambda_{\mu\nu} - {}^{(4)}K^\sigma_{\nu\gamma} {}^{(4)}K^\gamma_{\sigma\mu} = -\tilde{\Lambda}_4 q_{\mu\nu} + 8\pi G_N \pi_{\mu\nu} + k_5^4 Y_{\mu\nu} - \mathring{E}_{\mu\nu} \\ & + q_\mu^\alpha q_\nu^\beta \left[\frac{2}{3} \left(\nabla_{[\lambda} K^\lambda_{\beta\alpha]} + K^\sigma_{\gamma\sigma} K^\gamma_{\beta\alpha} - K^\lambda_{\beta\gamma} K^\gamma_{\lambda\alpha} \right) - n_\rho n^\sigma \left(\nabla_{[\sigma} K^\rho_{\alpha\beta]} + K^\rho_{\gamma[\sigma} K^\gamma_{\alpha\beta]} \right) \right], \end{aligned} \quad (68)$$

where

$$\begin{aligned} \tilde{\Lambda}_4 \equiv & \Lambda_4 - D^\lambda {}^{(4)}K^\tau_{\lambda\tau} + \frac{1}{2} {}^{(4)}K_{\tau\alpha}^\alpha {}^{(4)}K^{\tau\lambda}_\lambda - \frac{1}{2} {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} - \frac{2}{3} n^\alpha n^\beta \left(\nabla_\lambda K^\lambda_{\beta\alpha} - \nabla_\alpha K^\lambda_{\beta\lambda} \right. \\ & \left. + K^\lambda_{\gamma\lambda} K^\gamma_{\beta\alpha} - K^\sigma_{\beta\gamma} K^\gamma_{\sigma\alpha} \right) + \frac{1}{6} \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\alpha}^\alpha K^{\tau\lambda}_\lambda + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right). \end{aligned} \quad (69)$$

The function $\tilde{\Lambda}_4$ above is usually called effective cosmological “constant” in the literature, in the sense that all its terms are multiplied by the brane metric in the Einstein effective equation (68). Eqs. (68) and (69) show that the factors involving contortion, both in four and in five dimensions, modify drastically the effective Einstein equation on the brane and the effective cosmological constant as well.

Now, let us look at some deviations of the black hole horizon coming from the bulk torsion terms. Hereon in this Section we assume vacuum on the brane ($\pi_{\mu\nu} = 0 = Y_{\mu\nu}$) and neglect the contribution of the effective cosmological constant term, which is expected to be smaller, by some orders of magnitude, than the contribution of the term Weyl [10]. Using a Taylor expansion in the extra dimension in order to probe properties of a static black hole on the brane [23], the bulk metric can be written as

$$\begin{aligned} g_{\mu\nu}(x, y) = & q_{\mu\nu} - (\mathring{E}_{\mu\nu} + A_{\mu\nu}) y^2 - \frac{2}{l} (\mathring{E}_{\mu\nu} + A_{\mu\nu}) y^3 + \frac{1}{12} \left(\left(\square \mathring{E}_{\mu\nu} - \frac{32}{l^2} \mathring{E}_{\mu\nu} + 2\mathring{R}_{\mu\alpha\nu\beta} \mathring{E}^{\alpha\beta} + 6\mathring{E}_\mu^\alpha \mathring{E}_{\alpha\nu} \right) \right. \\ & \left. + \left(\square A_{\mu\nu} - \frac{32}{l^2} A_{\mu\nu} + 2(\nabla_{[\nu} K_{\mu\alpha\beta]} A^{\alpha\beta} + 2K_{\mu\gamma[\nu} K^\gamma_{\alpha\beta]} A^{\alpha\beta} + 6\mathring{A}_\mu^\alpha \mathring{A}_{\alpha\nu} \right) \right) y^4 + \dots \end{aligned}$$

where

$$\begin{aligned} A_{\mu\nu} = & \left(\nabla_{[\delta} K^\kappa_{\alpha\beta]} + K^\kappa_{\gamma[\beta} K^\gamma_{\alpha\delta]} \right) n_\kappa n^\delta q_\mu^\alpha q_\nu^\beta + \frac{1}{6} q_{\mu\nu} \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right) \\ & - \frac{2}{3} (q_\mu^\alpha q_\nu^\beta + n^\alpha n^\beta q_{\mu\nu}) \left(\nabla_{[\lambda} K^\lambda_{\beta\alpha]} + K^\lambda_{\gamma\lambda} K^\gamma_{\beta\alpha} - K^\sigma_{\beta\gamma} K^\gamma_{\sigma\alpha} \right) \end{aligned}$$

and \square denotes the usual d'Alembertian. As in [10], it shows in particular that the propagating effect of 5D gravity arises only at the fourth order of the expansion. For a static spherical metric on the brane given by

$$g_{\mu\nu} dx^\mu dx^\nu = -F(r) dt^2 + \frac{dr^2}{H(r)} + r^2 d\Omega^2, \quad (70)$$

the projected Weyl term on the brane is given by the expressions¹³

$$\begin{aligned}
E_{00} &= \frac{F}{r} \left(H' - \frac{1-H}{r} \right) + \left(\nabla_\nu K^{\mu 00} - \nabla_0 K^\mu_{0\nu} + K^\mu_{\gamma\nu} K^\gamma_{00} - K^\mu_{\gamma 0} K^\gamma_{0\nu} \right) n_\mu n^\nu F^2 \\
&\quad - \frac{2}{3} F(F-1) \left(\nabla_\lambda K^\lambda_{00} - \nabla_0 K^\lambda_{0\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{00} - K^\sigma_{0\gamma} K^\gamma_{\sigma 0} \right) + \frac{1}{6} F \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right), \\
E_{rr} &= -\frac{1}{rH} \left(\frac{F'}{F} - \frac{1-H}{r} \right) + \left(\nabla_\nu K^{\mu rr} - \nabla_r K^\mu_{r\nu} + K^\mu_{\gamma\nu} K^\gamma_{rr} - K^\mu_{\gamma r} K^\gamma_{r\nu} \right) n^\mu n^\nu H^{-2} \\
&\quad - \frac{2}{3} H^{-1} (H^{-1} - (n^r)^2) \left(\nabla_\lambda K^\lambda_{rr} - \nabla_r K^\lambda_{r\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{rr} - K^\sigma_{r\gamma} K^\gamma_{\sigma r} \right) \\
&\quad + \frac{1}{6} H^{-1} \left(2\nabla^\lambda K^\tau_{\lambda\tau} - K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma + K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right), \\
E_{\theta\theta} &= -1 + H + \frac{r}{2} H \left(\frac{F'}{F} + \frac{H'}{H} \right) + \left(\nabla_\nu K^{\mu\theta\theta} - \nabla_\theta K^\mu_{\theta\nu} + K^\mu_{\gamma\nu} K^\gamma_{\theta\theta} - K^\mu_{\gamma\theta} K^\gamma_{\theta\nu} \right) n_\mu n^\nu r^4 \\
&\quad - \frac{2}{3} r^2 (r^2 + 1) \left(\nabla_\lambda K^\lambda_{\theta\theta} - \nabla_\theta K^\lambda_{\theta\lambda} + K^\lambda_{\gamma\lambda} K^\gamma_{\theta\theta} - K^\sigma_{\theta\gamma} K^\gamma_{\sigma\theta} - \frac{1}{2} \nabla^\lambda K^\tau_{\lambda\tau} + \frac{1}{4} K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma - \frac{1}{4} K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right) \quad (71)
\end{aligned}$$

Note that in Eq.(70) the metric reduces to the Schwarzschild one, if $F(r)$ equals $H(r)$. The exact determination of these radial functions remains an open problem in black hole theory on the brane [10].

These components allow one to evaluate the metric coefficients in Eq.(70). The area of the 5D horizon is determined by $g_{\theta\theta}$. Defining $\psi(r)$ as the deviation from a Schwarzschild form H , i.e.,

$$H(r) = 1 - \frac{2M}{r} + \psi(r), \quad (72)$$

where M is constant, yields

$$\begin{aligned}
g_{\theta\theta}(r, y) &= r^2 + \psi' \left(1 + \frac{2}{l} y \right) + \left(\nabla_\nu K^{\mu\theta\theta} - \nabla_\theta K^\mu_{\theta\nu} + K^\mu_{\gamma\nu} K^\gamma_{\theta\theta} - K^\mu_{\gamma\theta} K^\gamma_{\theta\nu} \right) n_\mu n^\nu r^4 \\
&\quad - \frac{2}{3} r^2 (r^2 + 1) \left(\nabla_{[\lambda} K^\lambda_{\theta\theta]} + K^\lambda_{\gamma\lambda} K^\gamma_{\theta\theta} - K^\sigma_{\theta\gamma} K^\gamma_{\sigma\theta} - \frac{1}{2} \nabla^\lambda K^\tau_{\lambda\tau} + \frac{1}{4} K_{\tau\lambda}^\lambda K^{\tau\gamma}_\gamma - \frac{1}{4} K_{\tau\gamma\lambda} K^{\tau\lambda\gamma} \right) y^2 \\
&\quad + \dots \quad (73)
\end{aligned}$$

It shows how ψ and the contortion and its derivatives determine the variation in the area of the horizon along the extra dimension. Also, the variation in the black string properties can be extracted. Obviously, when the torsion goes to zero, all the results above are led to the ones obtained in [10], [23], and references therein. In particular, Eq.(70) — when the torsion, and consequently $A_{\mu\nu}$ defined in Eq.(70), goes to zero — is led to the results previously obtained in [10].

As the area of the a black hole 5D horizon is determined by $g_{\theta\theta}$, in particular it may indicate observable signatures of corrections induced by contortion terms, since for a given fixed effective extra dimension size, supermassive black holes give the upper limit of variation in luminosity of quasars. Also, it is possible to re-analyze how the quasar luminosity variation behaves as a function of the AdS₅ bulk radius — corrected by contortion terms — in some solar mass range, as in [24] and references therein.

Furthermore, braneworld measurable corrections induced by contortion terms for quasars, associated with Schwarzschild and Kerr black holes, by their luminosity observation are important. These corrections in a torsionless context were shown to be more notorious for mini-black holes, where the Reissner-Nordstrom radius in a braneworld scenario is shown to be around a hundred times bigger than the standard Reissner-Nordstrom radius associated with mini-black holes, besides mini-black holes being much more sensitive to braneworld effects. It is possible to repeat all the comprehensive and computational procedure in [24] in order to verify how the contortion effects in Eq.(73) can modify even more the above-mentioned results.

The modification in the area of the black hole horizon due to torsion terms, whose functional form is depicted in Eq. (73), can be better appreciated in a specific basis, i. e., an explicit *ansatz* for the spacetime metric. This is, however, out of the scope of the present work. The important point here is that torsion terms do affect the black hole horizon and the departure from the usual (torsionless) case is precisely given by Eq. (73). In the next Section we extend and apply the braneworld sum rules to the case with torsion, considering some estimates of the torsion effects.

¹³ In the three expressions below, the indices r and θ strictly denote the coordinates, and can not be confounded with summation indices.

VI. SUM RULES WITH TORSION

In this Section we shall derive the consistency conditions for braneworld scenarios embedded in a Riemann-Cartan manifold. The general procedure is quite similar to the one found in [25, 26] and we shall comprise some of the general formulation here, for the sake of completeness.

We start in a very general setup, analyzing a D -dimensional bulk spacetime geometry, endowed with a non-factorizable metric

$$\begin{aligned} ds^2 &= G_{AB} dX^A dX^B \\ &= W^2(r) g_{\alpha\beta} dx^\alpha dx^\beta + g_{ab}(r) dr^a dr^b, \end{aligned} \quad (74)$$

where $W^2(r)$ is the warp factor, X^A denotes the coordinates of the full D -dimensional bulk, x^α stands for the $(p+1)$ non-compact spacetime coordinates, and r^a labels the $(D-p-1)$ directions in the internal compact space. The D -dimensional Ricci tensor can be related to its lower dimensional partners by [25]

$$R_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{g_{\mu\nu}}{(p+1)W^{p-1}} \nabla^2 W^{p+1}, \quad (75)$$

$$R_{ab} = \tilde{R}_{ab} - \frac{p+1}{W} \nabla_a \nabla_b W, \quad (76)$$

where \tilde{R}_{ab} , ∇_a and ∇^2 are respectively the Ricci tensor, the covariant derivative, and the Laplacian operator constructed by means of the internal space metric g_{ab} . $\bar{R}_{\mu\nu}$ is the Ricci tensor derived from $g_{\mu\nu}$. Denoting the three curvature scalars by $R = G^{AB} R_{AB}$, $\bar{R} = g^{\mu\nu} \bar{R}_{\mu\nu}$, and $\tilde{R} = g^{ab} \tilde{R}_{ab}$ we have, from Eqs.(74) and (76),

$$\frac{1}{p+1} (W^{-2} \bar{R} - R^\mu_\mu) = p W^{-2} \nabla W \cdot \nabla W + W^{-1} \nabla^2 W \quad (77)$$

and

$$\frac{1}{p+1} (\tilde{R} - R^a_a) = W^{-1} \nabla^2 W, \quad (78)$$

where $R^\mu_\mu \equiv W^{-2} g^{\mu\nu} R_{\mu\nu}$ and $R^a_a \equiv g^{ab} R_{ab}$ ($R = R^\mu_\mu + R^a_a$). It can be easily verified that for an arbitrary constant ξ the following identity holds

$$\frac{\nabla \cdot (W^\xi \nabla W)}{W^{\xi+1}} = \xi W^{-2} \nabla W \cdot \nabla W + W^{-1} \nabla^2 W. \quad (79)$$

Combining the above equation with Eqs.(74) and (76) we have

$$\nabla \cdot (W^\xi \nabla W) = \frac{W^{\xi+1}}{p(p+1)} [\xi (W^{-2} \bar{R} - R^\mu_\mu) + (p-\xi) (\tilde{R} - R^a_a)]. \quad (80)$$

The D -dimensional Einstein equation is given by

$$R_{AB} = 8\pi G_D \left(T_{AB} - \frac{1}{D-2} G_{AB} T \right), \quad (81)$$

where G_D is the gravitational constant in D dimensions. It is easy to write down the following equations:

$$R^\mu_\mu = \frac{8\pi G_D}{D-2} (T^\mu_\mu (D-p-3) - T^m_m (p+1)), \quad R^m_m = \frac{8\pi G_D}{D-2} (T^m_m (p-1) - T^\mu_\mu (D-p-1)). \quad (82)$$

In the above equations we set $T^\mu_\mu = W^{-2} g_{\mu\nu} T^{\mu\nu}$ ($T^M_M = T^\mu_\mu + T^m_m$). Now, it is possible to relate R^μ_μ and R^m_m in Eq.(80) in terms of the stress-tensor. Note that the left hand side of Eq.(80) vanishes upon integration along a compact internal space. Hence, taking all that into account we have

$$\oint W^{\xi+1} \left(T^\mu_\mu [(p-2\xi)(D-p-1) + 2\xi] + T^m_m p (2\xi - p + 1) + \frac{D-2}{8\pi G_D} [(p-\xi)\tilde{R} + \xi \bar{R} W^{-2}] \right) = 0. \quad (83)$$

Let us to particularize the analysis for a 5-dimensional bulk, since it describes the phenomenologically interesting case. Besides, it makes the conclusions obtained here applicable to the case studied in [22], in continuity to the program

of developing formal concepts to braneworld scenarios with torsion. In this way $D = 5$, $p = 3$, and consequently $\tilde{R} = 0$, because there is just one dimension on the internal space. With such specifications and assuming implicitly, as usual, that the brane action volume element does not depend on torsion¹⁴, Eq.(83) becomes

$$\oint W^{\xi+1} \left(T^\mu_\mu + 2(\xi - 1)T^m_m + \frac{\xi}{\kappa_5^2} \bar{R} W^{-2} \right) = 0, \quad (84)$$

where $8\pi G_5 = \kappa_5^2 = \frac{8\pi}{M_5^3}$, with M_5 denoting the 5-dimensional Planck mass. In order to implement torsion terms in our analysis, the expressions for the Riemann and Ricci tensors in terms of contortion components related with their partners — constructed with the usual metric compatible Levi-Civita connection

$$\begin{aligned} \bar{R}^\lambda_{\tau\alpha\beta} &= \overset{\circ}{R}^\lambda_{\tau\alpha\beta} + \nabla_{[\alpha} {}^{(4)}K^\lambda_{\tau\beta]} + {}^{(4)}K^\lambda_{\gamma[\alpha} {}^{(4)}K^\gamma_{\tau\beta]} \\ \bar{R}_{\tau\beta} &= \overset{\circ}{R}_{\tau\beta} + \nabla_{[\lambda} {}^{(4)}K^\lambda_{\tau\beta]} + {}^{(4)}K^\lambda_{\gamma\lambda} {}^{(4)}K^\gamma_{\tau\beta} - {}^{(4)}K^\lambda_{\tau\gamma} {}^{(4)}K^\gamma_{\lambda\beta} \end{aligned} \quad (85)$$

where the label “(4)” on the contortion terms denotes the contortion of the 3-branes and the covariant derivative is considered when a connection that presents *no* torsion is taken into account. First, however, note that in order to reproduce the observable universe one can put $\tilde{R} = 0$ with $10^{-120} M_{Pl}$ of confidence level, where M_{Pl} is the Planck mass. Note that the observations concerning the scalar curvature are related to the torsionless \tilde{R} , not to \bar{R} . So, taking it into account it follows that

$$\oint W^{\xi+1} \left[T^\mu_\mu + 2(\xi - 1)T^m_m + \frac{\xi W^{-2}}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K^\tau_{\lambda\tau} - {}^{(4)}K^\lambda_{\tau\lambda} {}^{(4)}K^\tau_{\lambda\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \right] = 0. \quad (86)$$

In order to proceed with the consistency conditions we specify the standard *ansatz* for the stress-tensor. Assuming that there are no other types of matter in the bulk, except the branes and the cosmological constant, we have [26]

$$T_{MN} = -\frac{\Lambda}{\kappa_5^2} G_{MN} - \sum_i T_3^{(i)} P[G_{MN}]_3^{(i)} \delta(y - y_i), \quad (87)$$

where Λ is the bulk cosmological constant, $T_3^{(i)}$ is the tension associated to the i^{th} -brane and $P[G_{MN}]_3^{(i)}$ is the pull-back of the metric to the 3-brane. The partial traces of (87) are given by

$$T^\mu_\mu = \frac{-4\Lambda}{\kappa_5^2} - 4 \sum_i T_3^{(i)} \delta(y - y_i), \quad \text{and} \quad T^m_m = -\frac{\Lambda}{\kappa_5^2}, \quad (88)$$

in such way that Eq.(86) becomes

$$\oint W^{\xi+1} \left[\frac{2\Lambda}{\kappa_5^2} (\xi + 1) + 4 \sum_i T_3^{(i)} \delta(y - y_i) - \frac{\xi W^{-2}}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K^\tau_{\lambda\tau} - {}^{(4)}K^\lambda_{\tau\lambda} {}^{(4)}K^\tau_{\lambda\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \right] = 0. \quad (89)$$

As one can see, this formalism can be applied for a several branes scenario. The number of branes, nevertheless, is not so important to our analysis. To fix ideas let us particularize the formalism to the two branes case. Denoting $T_3^{(1)} = \lambda$, the visible brane, $T_3^{(2)} = \tilde{\lambda}$, and assuming that neither the cosmological constant nor the branes contortion terms do depend on the extra dimension, Eq.(89) gives

$$4\lambda W_\lambda^{\xi+1} + 4\tilde{\lambda} W_{\tilde{\lambda}}^{\xi+1} + \frac{2\Lambda}{\kappa_5^2} (\xi + 1) \oint W^{\xi+1} - \frac{\xi}{\kappa_5^2} \left(2D^\lambda {}^{(4)}K^\tau_{\lambda\tau} - {}^{(4)}K^\lambda_{\tau\lambda} {}^{(4)}K^\tau_{\lambda\lambda} + {}^{(4)}K_{\tau\gamma\lambda} {}^{(4)}K^{\tau\lambda\gamma} \right) \oint W^{\xi-1} = 0, \quad (90)$$

where $W_\lambda = W(y = y_1)$ and $W_{\tilde{\lambda}} = W(y = y_2)$. Now some physical outputs of the general Eq.(90) are analyzed, in order to investigate the viability of braneworld scenarios with torsion. The first case we shall look at relates a factorizable geometry. Nevertheless, before going forward, we shall emphasize that if one implements the torsion null case in Eq.(90), it is easy to see that for $\xi = -1$ one recovers the well known fine tuning of the Randall-Sundrum model, i.e.,

$$\lambda + \tilde{\lambda} = 0, \quad (91)$$

as expected.

¹⁴ In this way, we guarantee that the brane volume element reduces to d^4x in the limit of null torsion and flat space.

A. Non-warped compactifications with torsion

The non-warped case is implemented by imposing $W = 1$, working then in a factorizable spacetime geometry. The general approach on consistency conditions, as exposed before, allows this possibility. In this Subsection we are therefore concerned with the viability of braneworld scenarios in the general scope analyzed in reference [1] and in the presence of torsion. The case we are going to describe here is not the most interesting. We shall, however, study a little further this simplified case, since it can provide some physical insight to the warped case.

From Eq.(90), the non-warped case reads

$$\frac{2\Lambda}{\kappa_5^2}(\xi + 1)V + 4\lambda + 4\tilde{\lambda} - \frac{\xi V}{\kappa_5^2} \left(2D^\lambda ({}^{(4)}K_{\lambda\tau}^\tau - ({}^{(4)}K_{\tau\lambda}^\lambda ({}^{(4)}K_{\lambda}^{\tau\lambda} + ({}^{(4)}K_{\tau\gamma\lambda} ({}^{(4)}K^{\tau\lambda\gamma} \right) = 0, \quad (92)$$

where V denotes the “volume” of the internal space. Note that for $\xi = 0$, the torsion terms do not influence the general sum rules in the present case. In fact, for $\xi = 0$ it follows that

$$\frac{V\Lambda}{2\kappa_5^2} + \lambda + \tilde{\lambda} = 0, \quad (93)$$

which states that, for non-warped branes, it is possible to exist an AdS_5 bulk, even for strictly positive tension values associated with the branes. Another interesting case is obtained for $\xi = -1$. In such case the bulk cosmological constant is factored out and consequently

$$\left(({}^{(4)}K_{\tau\lambda}^\lambda ({}^{(4)}K_{\lambda}^{\tau\lambda} - ({}^{(4)}K_{\tau\gamma\lambda} ({}^{(4)}K^{\tau\lambda\gamma} - 2D^\lambda ({}^{(4)}K_{\lambda\tau}^\tau \right) = \frac{4\kappa_5^2}{V}(\lambda + \tilde{\lambda}). \quad (94)$$

Note that the left hand side (LHS) of (94) can be interpreted as the difference between $\overset{\circ}{R}$ and \bar{R} . In other words, the LHS of Eq.(94) measures the contribution of the torsion terms to the brane curvature, i. e., it indicates how much the brane curvature differs itself from zero, due to torsion terms. So, we can write schematically

$$\overset{\circ}{R} - \bar{R} = \frac{4\kappa_5^2}{V}(\lambda + \tilde{\lambda}). \quad (95)$$

We see that the effect of the torsion in the brane curvature is proportional to the branes tension values in the two branes scenario, but it decreases with the distance between the branes. Moreover, since $\kappa_5^2 = 8\pi G_5 \sim 1/M_5^3$, such an effect is about $1/(VM_5^3)$. Therefore, it indicates the low magnitude of torsion effects in the braneworld scenario with large extra transverse dimension, since it is suppressed by the 5-dimensional Planck scale and also by the volume of the internal space. Obviously, in a braneworld scenario which solves the hierarchy problem the typical scale of the higher dimensional Planck mass is of order $M_5 \sim M_{weak}$ and then, the suppression due to the internal space volume is attenuated.

B. The warped case

In the absence of a factorizable geometry, some configurations of the warp factor may be responsible for the right mass partition in the Higgs mechanism without the necessity of any additional hierarchy [3]. Starting from the general Eq.(90), we shall look at the most important cases, namely $\xi = -1, 0, 1$.

For $\xi = -1$ we have

$$\left(({}^{(4)}K_{\tau\lambda}^\lambda ({}^{(4)}K_{\lambda}^{\tau\lambda} - ({}^{(4)}K_{\tau\gamma\lambda} ({}^{(4)}K^{\tau\lambda\gamma} - 2D^\lambda ({}^{(4)}K_{\lambda\tau}^\tau \right) = \frac{4\kappa_5^2}{\oint W^{-2}}(\lambda + \tilde{\lambda}). \quad (96)$$

This is the warped analogue of Eq.(94) with the volume of the internal space replaced by the circular integral of W^{-2} in the denominator of the right hand side. The same conclusions as the $\xi = -1$ case of the previous Subsection still hold, but here we call the attention to the minuteness of the torsion terms: even contributing with such low magnitude effect to the brane curvature, it allows the branes to have both the same sign associated to their respective tension values.

The bulk spacetime type can be better visualized in the $\xi = 0$ case. Since all torsion terms of Eq.(90) are factored out, it follows that

$$\frac{\Lambda}{2\kappa_5^2} \oint W + \lambda W_\lambda + \tilde{\lambda} W_{\tilde{\lambda}} = 0. \quad (97)$$

Therefore, as $\oint W < 0$, it is easy to see that if $\lambda, \tilde{\lambda} > 0$ then necessarily $\Lambda > 0$ corresponding to an dS₅ bulk geometry. Otherwise, being $\lambda, \tilde{\lambda} < 0$ one arrives at an AdS₅ bulk geometry.

For the $\xi = 1$ case, a slight modification of Eq.(96) deserves a notification. The implementation of $\xi = 1$ in the Eq.(90) results in

$$\frac{\Lambda}{\kappa_5^2} \oint W^2 + \lambda W_\lambda^2 + \tilde{\lambda} W_{\tilde{\lambda}}^2 - \frac{V}{4\kappa_5^2} \left(2D^\lambda ({}^{(4)}K_{\lambda\tau}^\tau - ({}^{(4)}K_{\tau\lambda}^\lambda ({}^{(4)}K_{\lambda\lambda}^{\tau\lambda} + ({}^{(4)}K_{\tau\gamma\lambda} ({}^{(4)}K^{\tau\lambda\gamma})) \right) = 0. \quad (98)$$

Now, isolating the torsion contribution to the curvature we have

$$\left(2D^\lambda ({}^{(4)}K_{\lambda\tau}^\tau - ({}^{(4)}K_{\tau\lambda}^\lambda ({}^{(4)}K_{\lambda\lambda}^{\tau\lambda} + ({}^{(4)}K_{\tau\gamma\lambda} ({}^{(4)}K^{\tau\lambda\gamma})) \right) = \frac{4\Lambda}{V} \oint W^2 + \frac{4\kappa_5^2}{V} (\lambda W_\lambda^2 + \tilde{\lambda} W_{\tilde{\lambda}}^2). \quad (99)$$

From Eq.(99) we see that the torsion contribution to the brane curvature is constrained by the internal space volume, however terms coming from the warped compactification — as $\oint W^2$ and $\oint W_{\lambda,\tilde{\lambda}}^2$ — can turn this contribution more appreciable. In particular, the first term of the right hand side of (99) is the dominant one, since it is not suppressed by the 5-dimensional Planck scale and it is multiplied by the bulk cosmological constant. We shall make more comments about these results in the next Section.

VII. CONCLUDING REMARKS AND OUTLOOK

There are some alternative derivations of the junction conditions for a brane in a 5-dimensional bulk, when Gauss-Bonnet equations are used to describe gravity [27]. Also, Israel junction conditions can be generalized for a wider class of theories by direct integration of the field equations, where a specific non-minimal coupling of matter to gravity suggests promising classes of braneworld scenarios [28]. In addition, it is also possible to generalize matching conditions for cosmological perturbations in a teleparallel Friedmann universe, following the same lines as [29].

In the case studied here, however, the matching conditions are not modified by the inclusion of torsion terms in the connection. As noted, it is a remarkable and unexpected characteristic. Besides, all the development concerning the formalism presented is accomplished in the context of braneworld models. In such framework, the appearance of torsion terms is quite justifiable. However, the fact that the matching conditions remain unalterable in the presence of torsion is still valid in usual 4-dimensional theories.

Once investigated the junction conditions, we have obtained, via Gauss-Codazzi formalism, the Einstein effective projected equation on the brane. If, on one hand, the torsion terms do not intervenes in the usual Israel-Darmois conditions, on the other hand it modifies drastically the brane Einstein equations. Eq.(69) shows up the strong dependence of the new effective cosmological constant on the four and five-dimensional contorsion terms. It reveals promising possibilities. For instance, by a suitable behavior of such new terms, $\tilde{\Lambda}_4$ can be very small. In a more complete scenario, $\tilde{\Lambda}_4$ could be not even a constant. It must be stressed that these types of modification in the projected Einstein equation also appear in other models in modified gravity [31].

This Chapter intends to give the necessary step in order to formalize the mathematical implementation of torsion terms in braneworld scenarios. The application of our results are beyond the scope of this work. We finalize, however, pointing out some interesting research lines coming from the use of the results — obtained in this work — in cosmological problems.

The final result is very important from the cosmological viewpoint. It is clear that deviations of the usual braneworld cosmology can be obtained from the analysis of phenomenological systems in the light of Eq.(68). Physical aspects, more specifically the analysis of cosmological signatures as found in ref. [32], arising from the combination of the extra dimensions and torsion should be systematically investigated and compared with usual braneworld models. The ubiquitous presence of torsion terms leads, by all means, to subtle but important deviations of usual braneworlds in General Relativity. For instance, the equation (68) can be used as a starting point to describe the flat behavior of galaxy rotational curves without claim for dark matter. This last problem was already analyzed in the context of brane worlds [33], however the outcome arising from the torsion terms has never been investigated. A systematic comparative study between usual braneworld models and those braneworld models embedded in an Einstein-Cartan manifold is, potentially, interesting since it can lead us to new branches inside brane physics. We shall address to those questions in the future.

Two additional remarks must be pointed out. First, all the development concerning the formalism presented is accomplished in the context of braneworld models. In such framework, the appearance of torsion terms is quite justifiable. However, the fact that the matching conditions remain unalterable in the presence of torsion is still valid in usual 4-dimensional theories. Second, the discontinuity orthogonal to the brane is analyzed, since it is the unique possibility: the brackets $[A]$ of any quantity A denote, by definition, the jump across the brane. The geometric

reason points in the same direction; the connection must be continuous along the brane in order to guarantee the full applicability of standard calculations on the brane that works as a model to the universe.

This work concerns some effects evinced by torsion terms corrections — both in the bulk and on the brane. To study a typical gravitational signature arising from a gravitational system we performed in Section II the analysis based upon the well known Taylor expansion tools — strongly reminiscent of the assumption of a direction orthogonal to the brane — of the bulk metric in terms of the brane metric, taking into account bulk torsion terms. Our main result is summarized by Eq.(73). It shows how the bulk torsion terms intervene in the black hole area, in an attempt to find some observable effects arising from the torsion properties. Again, its highly non-trivial form can be better studied in the context of a specific model. It is out of the scope of this paper, nevertheless we shall point a line of research in this area. It could be interesting to apply the results found in this Section to some gravitational systems, in analogy to what was accomplished in standard braneworld scenarios (see, for instance, references [24]).

In order to study the behavior of the brane torsion terms we extend, in Section IV, the braneworld sum rules. It was demonstrated that the consistency conditions do not preclude the possibility of torsion on the brane. It was shown, however, that the torsion effects in the brane curvature are suppressed. It could, in principle, explain a negative result for experiments from the geometrical point of view. Just for a comparative complement, in reference [21] the 5-dimensional torsion field, identified with the rank-2 Kalb-Ramond (KR) field, was considered in the bulk. It was demonstrated by the authors the existence of an additional exponential damping for the zero-mode of the KR field arisen from the compactification of the transverse dimension. In some sense, our purely geometrical sum rules complete the analysis concerning the presence of torsion, this time on the brane.

In this paragraph we would like to call attention for some related issues appearing in the literature. In [21] it was shown that in an effective 4-dimensional theory on the visible brane, the KR field — as a source of torsion — is suppressed when a torsion-dilaton-gravity action in a Randall-Sundrum braneworld scenario is considered, explaining the apparent insensitivity of torsion in the brane. It was shown, however, that even in this case the KR field may led to new signatures in TeV scale experiments, when a coupling between dilaton and torsion is taken into account. The warped extra-dimensional formalism points to the presence of new interactions, of significant phenomenological importance, between the Kaluza-Klein modes of the dilaton and the KR field.

Briefly speaking, the results of this paper point to the fact that the hypothesis of a torsionless brane universe may be based upon a justified impression, since its effects from the bulk (studied from a quantum field theory approach) and from the brane (analyzed via the geometrical sum rules) are suppressed by some damping factor. We emphasize, however, that in the context above, the naive estimative of the 4-dimensional torsion effects (99) must be complemented by the results of a more specific system via Eq. (73). Such a characterization may put this gravitational and geometrical approach in the same level, concerning the brane torsion phenomenology, as, for instance, the massive spectrum of 5-dimensional KR field signature which can be viewed in a TeV-scale accelerator [21]. In this vein, the torsionless brane universe may be naturally substituted by a more fidedigne braneworld scenario that contains torsion, and may be useful to a more precise description of physical theories.

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- [1] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B 125** (1983) 136-138; V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. **B 125** (1983) 139-143; N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 429** (1998) 263-272 [[arXiv:hep-ph/9803315v1](#)]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 436** 257-263 (1998) [[arXiv:hep-ph/9804398v1](#)]; N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Rev. **D 59** (1999) 086004 [[arXiv:hep-ph/9807344v1](#)].
 - [2] P. Horava and E. Witten, Nucl. Phys. **B 460** (1996) 506-524 [[arXiv:hep-th/9510209v2](#)].
 - [3] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, (1999) 3370-3373 [[arXiv:hep-ph/9905221v1](#)]; L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 4690-4693 [[arXiv:hep-th/9906064v1](#)].
 - [4] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, Phys. Rev. **D 75** (2007) 084028 [[arXiv:hep-th/0703234v1](#)]; M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, [[arXiv:0707.0233 \[hep-th\]](#)].
 - [5] R. M. Wald, General Relativity, University of Chicago Press, 1984.
 - [6] T. Shiromizu, K. Maeda, and M. Sasaki, Phys. Rev. **D 62** (2000) 043523 [[arXiv:gr-qc/9910076v3](#)].
 - [7] P. Bostock, R. Gregory, I. Navarro, and J. Santiago, Phys. Rev. Lett. **92** (2004) 221601 [[arXiv:hep-th/0311074v2](#)].
 - [8] W. Israel, Nuovo Cimento **44B** (1966) 1.
 - [9] P. MacFadden, PhD thesis (2006) [[arXiv:hep-th/0612008v2](#)], Appendix A.
 - [10] R. Maartens, *Brane-world gravity*, Living Rev. Relativity **7**, 7 (2004) [[arXiv:gr-qc/0312059v1](#)].

- [11] Poincaré, H., *Hypothesis and Science*¹⁵, Dover Publ. Inc., New York, 1952.
- [12] Rodrigues, W. A. Jr. and Capelas de Oliveira, E., *The Many Faces of Maxwell, Dirac and Einstein Equations. A Clifford Bundle Approach.*, Lecture Notes in Physics **722**, Springer, Heidelberg, 2007.
- [13] Mosna, R. A., and Rodrigues, W. A. , Jr., The Bundles of Algebraic and Dirac-Hestenes Spinor Fields, *J. Math. Phys.* **45**, 2945-2966 (2004). [[math-ph/0212033](#)].
- [14] Thirring, W., and Wallner, R., The Use of Exterior Forms in Einstein's Gravitational Theory, *Brazilian J. Phys.* **8**, 686-723 (1978).
- [15] T. Frankel, *The Geometry of Physics: an Introduction*, 2nd ed., Cambridge Univ. Press, Cambridge, 2004.
- [16] M. Nakahara, *Geometry, Topology and Physics*, 2nd ed., Institute of Physics Publishing, Bristol, 2004.
- [17] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. I, Interscience Publishers, New York, 1963.
- [18] R. A. Mosna and W. A. Rodrigues, Jr., *J. Math. Phys.* **45** (2004) 2945-2966 [[arXiv:math-ph/0212033v5](#)].
- [19] W. A. Rodrigues, Jr., R. da Rocha, and J. Vaz, Jr., *Int. J. Geom. Meth. Mod. Phys.*, **2** (2005) 305-357 [[math-ph/0501064v6](#)].
- [20] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge 1987.
- [21] B. Mukhopadhyaya, S. Sen, and S. SenGupta, *Phys. Rev. Lett.* **89**, 121101 (2002); Erratum-ibidem **89**, 259902 (2002) [[arXiv:hep-th/0204242v2](#)].
- [22] J. M. Hoff da Silva and R. da Rocha, *Class. Quant. Grav.* **26**, 055007 (2009) [[arXiv:0804.4261v4](#) [[gr-qc](#)]]; Corrigendum ibidem **26** (2009) 179801.
- [23] N. Dadhich, R. Maartens, P. Papadopoulos, and V. Rezanian, *Black holes on the brane*, *Phys. Lett.* **B487**, 1-6 (2000) [[arXiv:hep-th/0003061v3](#)].
- [24] R. da Rocha and C. H. Coimbra-Araújo, *Phys. Rev. D* **74** (2006) 055006 [[arXiv:hep-ph/0607027v3](#)]; *J. Cosmol. Astropart. Phys.* **12** (2005) 009 [[arXiv:astro-ph/0510318v2](#)].
- [25] G. Gibbons, R. Kallosh, and A. Linde, *JHEP* **0101**, 022 (2001) [[arXiv:hep-th/0011225v2](#)].
- [26] F. Leblond, R. C. Myers, and D. J. Winters, *JHEP* **0107**, 031 (2001) [[arXiv:hep-th/0106140v2](#)].
- [27] N. Deruelle and C. Germani, *Nuovo Cim.* **118B** (2003) 977-988 [[arXiv:gr-qc/0306116v1](#)].
- [28] N. Deruelle, M. Sasaki, and Y. Sendouda, (2007) [[arXiv:0711.1150v1](#) [[gr-qc](#)]].
- [29] N. Deruelle and V. F. Mukhanov, *Phys. Rev. D* **52** (1995) 5549-5555 [[arXiv:gr-qc/9503050v1](#)].
- [30] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, *Phys. Rev. D* **75** (2007) 084028 [[arXiv:hep-th/0703234v1](#)]; M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, [[arXiv:0707.0233](#) [[hep-th](#)]].
- [31] M. C. B. Abdalla, M. E. X. Guimarães, and J. M. Hoff da Silva, *Eur. Phys. J. C* **55** (2008) 337-342 [[arXiv:0711.1254v2](#) [[hep-th](#)]].
- [32] R. da Rocha and C. H. Coimbra-Araújo, *JCAP* **0512** (2005) 009 [[arXiv:astro-ph/0510318v2](#)]; Roldao da Rocha and Carlos H. Coimbra-Araújo, *Phys. Rev. D* **74** (2006) 055006 [[arXiv:hep-ph/0607027v3](#)].
- [33] M. K. Mak and T. Harko, *Phys. Rev D* **70**, 024010 (2004) [[arXiv:gr-qc/0404104v1](#)]; F. Rahaman, M. Kalam, A. DeBenedictis, A. A. Usmani, and Saibal Ray, *Mon. Not. Roy. Astron. Soc.* **389**, 27 (2008) [[arXiv:0802.3453v2](#) [[astro-ph](#)]].

¹⁵ First published in English in 1905 by Walter Scott Publishing Co., Ltd.